Chapter 1
Notes

# 1 Definitions

**Definition 1.** A *linear equation* in the variables $x_1, x_2, \ldots x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \ldots a_n x_n = b$$

Where $b, a_1, a_2, \ldots a_n$ are real or complex numbers. We call $a_1, a_2, \ldots a_n$ the *coefficients*.

**Definition 2.** A *system of linear equations* (or linear system) is a collection of one or more linear equations involving the same variables.

**Definition 3.** A *solution* of a system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that make each equation true when $s_1, s_2, \ldots, s_n$ is substituted for $x_1, x_2, \ldots, x_n$.

**Definition 4.** The set of all possible solutions is called the *solution set* of a linear system. Two systems are *equivalent* if they have the same solution set.

**Definition 5.** A system is said to be *consistent* if it has solutions and *inconsistent* if it has no solutions.

**Definition 6.** Elementary Row Operations

(i) Replace - replace one row with the sum of itself and a multiple of another.

(ii) Interchange - switch the position of two rows.

(iii) Scale - multiply all terms of a row by a non-zero constant.

**Definition 7.** Two matrices are called *row equivalent* if there is a sequence of elementary row operations that transform one matrix into the other.

**Definition 8.** If a matrix $A$ is row equivalent to a matrix $U$ in row echelon form we call $U$ an *echelon form* of $A$. If a matrix $A$ is row equivalent to a matrix $U$ in reduced row echelon form we call $U$ the *reduced row echelon form* of $A$.

**Definition 9.** A row (or column) is *non-zero* if there is one non-zero entry

**Definition 10.** A *leading entry* of a row is the leftmost non-zero entry (in a non-zero row).

**Definition 11.** A rectangular matrix is in *row echelon form* (REF) if it has the following three properties

(i) All non-zero rows are above rows of all zero's.

(ii) Each leading entries of a row is in a column to the right of the leading entry of the row above it.

(iii) All entries in a column <u>below</u> a leading entry are zeros

If a REF matrix satisfies the following it is in *reduced row echelon form* (RREF)

(i) The leading entry in each non-zero row is a 1.

(ii) Eaching leading 1 is the only non-zero entry in its column.

**Example 1.** $\square$- nonzero entry, $*$- any real number

$$\begin{bmatrix} 0 & \square & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \square & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \square & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \square & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & * \end{bmatrix} \quad \text{(Row Echelon Form)}$$

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \quad \text{(Reduced Row Echelon Form)}$$

**Definition 12.** A *pivot position* in a matrix $A$ is a location in $A$ that corresponds to a leading 1 in the RREF of $A$.

**Definition 13.** A *pivot column* in a matrix $A$ is the column that contains the pivot position.

**Definition 14.** A *pivot* is a non-zero entry used to make 0's via row operations.

**Definition 15.** A *basic variable* is a variable corresponding to a pivot column. All other variables are *free variables*

**Definition 16.** Given vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, \ldots, c_p$, the vector $y$ defined by

$$y = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

is called a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with *weights* $c_1, \ldots c_p$.

**Definition 17.** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is called the *span* of the set of vectors and denoted $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

**Definition 18.** A system of linear equations is said to be *homogeneous* if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where $A$ is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$. The equation $A\mathbf{x} = \mathbf{0}$ always has at least one solution $\mathbf{x} = \mathbf{0}$ called the *trivial solution.* A *non-trivial solution* is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

**Definition 19.** An indexed set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be *linearly independent* of the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = 0$$

has only the trivial solution $x_1 = x_2 = \ldots = x_p = 0$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be *linearly dependent* of there are weights $c_1, \ldots, c_p$ not all zero such that

$$c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = 0 \tag{1}$$

Equation (1) is called a *linear dependence relation* among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

**Definition 20.** A *transformation* $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the *domain* of $T$ and $\mathbb{R}^m$ is called the *codomain* of $T$. The set of images $T(\mathbf{x})$ is called the *range* of $T$ and is a subset of the codomain.

**Definition 21.** A transformation $T$ is called *linear* if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of $T$

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars $c$ and all $\mathbf{u}$ in the domain of $T$.

**Definition 22.** A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *onto* $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of atleast one $\mathbf{x}$ in $\mathbb{R}^n$.

**Definition 23.** A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one to one* if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$ (could be none).

## 2 Propositions and Theorems

**Proposition 1.** *In general a system of linear equations has*

*1) No solutions*

*2) Exactly one solution*

*3) infinitely many solutions.*

**Proposition 2.** *If the augmented matrices of two linear systems are row equivalent they have the same solution set.*

**Theorem 3.** *Each matrix is row equivalent to one and only one reduced row echelon matrix.*

**Theorem 4.** *A linear system is consistent if and only if the rightmost columns of the augmented matrix is* <u>*not*</u> *a pivot column, i.e. if and only if an echelon form of the matrix has no row of the form*

$$[0, \ldots, 0, b]$$

*If a linear system is consistent then the solution set contains either*

*(i) a unique solution (no free variables)*

*(ii) infinite solutions (at least* 1 *free variable)*

**Proposition 5.** *A vector equation*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_n\mathbf{a}_n = \mathbf{b}$$

*has the same solution set as the linear system whose augmented matrix is*

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n \ \mathbf{b}]. \tag{2}$$

*In particular* $\mathbf{b}$ *can be generated by a linear combination of* $\mathbf{a}_1, \ldots \mathbf{a}_n$ *if and only if there exists a solution to the system with matrix (2)*

**Theorem 6.** *If $A$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots \mathbf{a}_n$ and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation*

$$A\mathbf{x} = \mathbf{b}$$

*has the same solution set as the vector equation*

$$x_1\mathbf{a}_1 + \dots x_n\mathbf{a}_n = \mathbf{b}$$

*which in turn has the same solution set as the system of linear equations with augmented matrix*

$$[\mathbf{a}_1 \ \dots \ \mathbf{a}_n \ b]$$

**Proposition 7.** *The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of the matrix $A$.*

**Theorem 8.** *Let $A$ be an $m \times n$ matrix. The following statements are logically equivalent.*

*(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*

*(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.*

*(c) The columns of $A$ span $\mathbb{R}^m$.*

*(d) $A$ has a pivot position in every row.*

**Proposition 9.** *The homogeneous equation $A\mathbf{x} = 0$ has a non-trivial solution if and only if the equation has atleast one free variable.*

**Theorem 10.** *Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. The the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form*

$$\mathbf{w} = \mathbf{p} + \mathbf{v}_h$$

*where $v_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.*

**Proposition 11.** *The columns of a matrix $A$ are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has* **only** *the trivial solution.*

**Theorem 12.** *An indexed set $S = \{\mathbf{v}_1, \dots \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$ then some $\mathbf{v}_j$ with $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$*

**Theorem 13.** *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1 \dots \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly independent if $p > n$.*

**Theorem 14.** *If a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.*

**Theorem 15.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There there exists a unique matrix $A$ such that*

$$T(\mathbf{x}) = A\mathbf{x} \textit{ for all } \mathbf{x} \textit{ in } \mathbb{R}^n.$$

*In fact, $A$ is the $m \times n$ matrix whose $j$th column is the vector $T(\mathbf{e_j})$ where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$.*

$$A = [T(\mathbf{e}_1) \ \ldots \ T(\mathbf{e}_n)]$$

*We call $A$ the* **standard matrix** *for $T$.*

**Theorem 16.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one to one if and only if the the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution*

**Theorem 17.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with $A$ as its standard matrix. Then:*

*(a) $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $A$ span $\mathbb{R}^m$.*

*(b) $T$ is one to one if and only if the columns of $A$ are linearly independent.*

# 3 Solving Equations

Here's an example on how to write a linear system in several equivalent ways.

$$\begin{array}{rcrcrl} x_1 & - & 2x_2 & + & x_3 & = 0 \\ & & 2x_2 & - & 8x_3 & = 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = -9 \end{array} \iff \left[\begin{array}{rrr:r} 1 & -2 & 1 & 0 \\ 0 & -1 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \text{(Augmented matrix)}$$

$$\iff \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & -8 \\ -4 & 5 & 9 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix} \text{(Matrix equation)}$$

$$\iff x_1 \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix} \text{(Vector Equation)}$$

## Row Reduction Algorithm:

**Forward phase:**

1) Begin with the left most non-zero column. This is a pivot column. The pivot position is at the top.

2) Select a non-zero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

3) Use row replacement operators to create zeros in all positions below the pivot

4) Ignore the row containing the pivot and cover all rows (if any) above it. Apply steps 1-3 on the remaining submatrix. Repeat the process untill there are no more non-zero rows to modify.

**Backwards phase:**

5) Beginning with the rightmost pivot an working upward and to the left, create zeros above each pivot. If a pivot is not a 1, make it 1 by scaling.

## Solving a Linear system:

1) Write the augmented matrix.

2) Use row reduction to reduce to row echelon form. Decide if system is consistent.

3) Go onto reduced row echelon form.

4) Write system for RREF.

5) Rewrite in parametric description.

# 4 Examples

**Example 2.** Let's solve this system

$$\begin{array}{rcrcrl} x_1 & + & 2x_2 & + & 3x_3 & = 9 \\ 2x_1 & - & x_2 & + & x_3 & = 8 \\ 3x_1 & & & - & x_3 & = 3 \end{array}$$

It is equivalent to the following augmented matrix

$$\left[\begin{array}{rrr:r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

We do the forward phase of row reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

At this point if we convert back into a linear system we can see the system is consistent.

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = 9 \\ & & x_2 & + & x_3 & = 2 \\ & & & & x_3 & = 3 \end{array}$$

At this point we can see that we have a solution and that we have no free variables. We can go further and use the backwards phase of row reduction to get the reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The system now reduces to

$$\begin{array}{cccc} x_1 & & & = 2 \\ & x_2 & & = -1 \\ & & x_3 & = 3 \end{array}$$

This is the only solution since there are no free variables. We also have the solution for the corresponding matrix and vector equation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix} \text{ has solution } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

That is,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix} \text{ has solution } (x_1, x_2, x_3) = (2, -1, 3)$$

That is,

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

**Example 3.** More generally, we consider the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Since

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We know by Theorem 8 that the matrix equation has a solution for every possible vector $\mathbf{b}$.

**Example 4.** Consider the homogeneous equation,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

then the homogeneous equation has only the trivial solution by Theorem 9.

**Example 5.** Consider the following linear transformation.

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \cdot \mathbf{x}$$

We know this transformation is one to one and onto by the use of Theorems 7, 8, 9, 10, 11, and 17.