MATH2210Q

Name: Soln

Practice Final

Date:

This exam contains 10 pages (including this cover page) and 12 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
9	0	
10	0	
11	0	
12	0	
Total:	0	

1. (points) Determine if the following set of vectors is orthogonal.

$$\left\{ \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}}_{= \vec{u}_1}, \underbrace{\begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}}_{= \vec{u}_2}, \underbrace{\begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}}_{= \vec{u}_3} \right\}$$

$\vec{u}_1 \cdot \vec{u}_2 = -3 - 6 - 3 + 12 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = 9 - 16 + 7 + 0 = 0$

$\vec{u}_2 \cdot \vec{u}_3 = -3 + 24 - 21 + 0 = 0$

2. (a) Show that $\mathbf{u}_1$ and $\mathbf{u}_2$ is an orthogonal basis for $\mathbb{R}^2$.

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$\dim(\mathbb{R}^2) = 2$ $\qquad$ $\vec{u}_1 \cdot \vec{u}_2 = 12 - 12 = 0$

so $\vec{u}_1$ and $\vec{u}_2$ are orthogonal

hence linearly indep.

$\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbb{R}^2$

(b) Express the vectors below as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

$$\mathbf{x}_1 = \begin{bmatrix} 9 \\ -7 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\vec{u}_1 \cdot \vec{u}_1 = 4 + 9 = 13$

$\vec{u}_2 \cdot \vec{u}_2 = 36 + 16 = 52$

$\vec{x}_1 \cdot \vec{u}_1 = 18 + 21 = 39$

$\vec{x}_1 \cdot \vec{u}_2 = 54 - 28 = 26$

$\vec{x}_2 \cdot \vec{u}_1 = 2 - 6 = -4$

$\vec{x}_2 \cdot \vec{u}_2 = 6 + 8 = 14$

$\vec{x}_1 = \left(\frac{39}{13}\right) \vec{u}_1 + \frac{26}{52} \vec{u}_2$

$\vec{x}_2 = \left(\frac{-4}{13}\right) \vec{u}_1 + \left(\frac{14}{52}\right) \vec{u}_2$

3. Consider the set of vectors below.

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$$

(a) (points) Let $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ compute the orthogonal projection of $\mathbf{y}$ onto W.

$\vec{y} \cdot \vec{u}_1 = -3 - 2 + 12 = 7$

$\vec{u}_1 \cdot \vec{u}_1 = 9 + 1 + 4 = 14$

$\vec{y} \cdot \vec{u}_2 = -1 - 2 - 12 = -15$

$\vec{u}_2 \cdot \vec{u}_2 = 1 + 1 + 4 = 6$

$\text{proj}_W \vec{y} = \hat{y} = \left(\frac{7}{14}\right) \vec{u}_1 + \left(\frac{-15}{6}\right) \vec{u}_2$

$\hat{y} = \begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix} - \begin{bmatrix} 5/2 \\ -5/2 \\ -5 \end{bmatrix}$

$\hat{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$

(b) (points) Compute the distance between $\mathbf{y}$ and W.

$\text{dist}(\vec{y}, W) = \| y - \hat{y} \| = \left\| \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\| = 0$

4. Let,

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

and $W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Find an orthogonal basis for W.

$\vec{V}_1 = \vec{U}_1$

$$\vec{V}_2 = \vec{U}_2 - \frac{\vec{U}_1 \cdot \vec{U}_2}{\vec{U}_1 \cdot \vec{U}_1} \vec{U}_1$$

$$\Rightarrow \vec{V}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{30}\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

Scaling V_2: $\vec{V}_2' = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$

Note $\vec{V}_1 \cdot \vec{V}_2' = 12 - 15 + 3 = 0$

Orthogonal basis for W

$$\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} \right\}$$

5. (points) Find the solution set to the system of equations below.

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + 2x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Solution set: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}$

6. (points) Determine if the columns of the matrix below form a linearly independent set.

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 5 \\ 1 & 1 & -5 \\ 2 & 1 & -10 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \\ 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 & 0 \\ 2 & 1 & -10 & 0 \\ -4 & -3 & 0 & 0 \\ 0 & -1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -20 & 0 \\ 0 & -1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -20 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \iff \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$A\vec{x} = \vec{0}$ has only the trivial soln

So the columns of A are linearly independent.

7. (points) Find a basis for both Col(A) and Null(A) for the matrix below.

$$A = \begin{bmatrix} 1 & -2 & -1 & 5 & 4 \\ 2 & -1 & 1 & 5 & 6 \\ -2 & 0 & -2 & 1 & -6 \\ 3 & 1 & 4 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Null(A) :

$$\begin{bmatrix} 1 & -2 & -1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= x_3 \\ x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

Basis for Null (A)

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Col(A) : pivot columns : 1, 2, 4, 5

Basis for Col(A)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -6 \\ 5 \end{bmatrix} \right\}$$

8. Let

$$\mathbf{b}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

and note that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are bases for $\mathbb{R}^2$.

(a) (points) Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

$$\left[\begin{array}{cc:cc} 2 & -2 & 4 & 8 \\ 2 & 2 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & -1 & 2 & 4 \\ 1 & 1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & -1 & 2 & 4 \\ 0 & 2 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 0 & 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$$P_{C \leftarrow B} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

(b) (points) Find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$.

$$\det \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} = -2$$

$$P_{B \leftarrow C} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \end{bmatrix}$$

(c) (points) Let $[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $[x]_{\mathcal{C}}$.

$$\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$[X]_C = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

9. Find a basis for the eigenspace for each of the eigenvalues below.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}, \quad \lambda = 4, \lambda = 3$$

Eigenspace for $\lambda = 4$:

$$[A - 4I \mid 0] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -3 & 0 & 9 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \iff \begin{array}{l} x_1 = 0 \\ x_2 = \text{free} \\ x_3 = 0 \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Eigenspace for $\lambda = 3$:

$$[A - 3I \mid 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \iff \begin{array}{l} x_1 = \frac{1}{3} x_2 + 3 x_3 \\ x_2 = \text{free} \\ x_3 = \text{free} \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

10. (a) (points) Calculate the rank and nullity of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns: 1 & 2

rank(A) = 2 nullity(A) = 2

(b) (points) Is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ onto?

$T: \mathbb{R}^4 \to \mathbb{R}^3$ and $\dim(\mathrm{Col}(A)) = 2$

So T is <u>not</u> onto.

(c) (points) Is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ one to one?

No, nullity(A) = 2.

11. (points) Find the matrix for the following linear transformation.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 5x_2 \\ x_2 - x_3 \\ x_1 + x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

12. Calculate the determinant of A^3 where,

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 5 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\det A = 1 \cdot \det \begin{bmatrix} -2 & 5 \\ 2 & 2 \end{bmatrix} = -14$$

$$\det A^3 = (-14)^3$$