MATH 3150
Homework 1 Solution

1. Let $\mathbb{C}$ denote the complex numbers with the standard addition and multiplication. Show that there is no order relation $>$ such that $\mathbb{C}$ is an ordered field. As a reminder:

Definition 0.1. An *ordered field* $\mathbb{F} = (\mathbb{F}, +, \cdot, <)$ consists of a field $(\mathbb{F}, +, \cdot)$ together with a relation $<$ on $\mathbb{F}$, called an *order*, satisfying

(i) (*trichotomy*) for each $x, y \in S$, exactly one of the following hold,

$$x < y, \quad y < x, \quad x = y;$$

(ii) (*transitivity*) for $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

(iii) if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$;

(iv) if $x, y \in \mathbb{F}$ and $x, y > 0$, then $xy > 0$.

Solution: We begin by stating and proving a lemma.

Lemma 0.2. *If $x \in \mathbb{F}$ where $\mathbb{F}$ is an ordered field, then $x^2 \geq 0$.*

Proof. Suppose that x is in the ordered field $\mathbb{F}$. By the trichotomy we have either $x > 0$, $x < 0$ or $x = 0$. If $x = 0$ then the result is clear. Now suppose $x > 0$, by the ordered field axioms we have that $x^2 = x \cdot x > 0$. If $x < 0$ then we have that $-x > 0$ and that $x^2 = (-x) \cdot (-x) > 0$. □

We can now prove our main result.

Proof. Suppose there is an order relation $<$ on $\mathbb{C}$ that turns $\mathbb{C}$ into an ordered field. Consider the element $i \in \mathbb{C}$. By trichotomy we have either $i > 0$, $i < 0$ or $i = 0$. Moreover, $i^2 = (-i)^2 = -1$ which violates our above lemma. Hence, no such order relation $<$ can exist turning $\mathbb{C}$ into an ordered field. □

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2. Find the supremum and infimum of the following set: $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Prove your claim.

Solution: We will show that $\sup(S) = 1$ and $\inf(S) = 0$.

a) $1 = \sup(S)$.

Proof. To show that 1 is the supremum of S we will first show that it is an upper bound and then show $1 - \varepsilon$ for any $\varepsilon > 0$ is no longer an upper bound. Note that $S = \{\frac{1}{n} : n \in \mathbb{N}^+\}$ and that $1 - \frac{1}{n} = \frac{n-1}{n} \geq 0$ for any $n \in \mathbb{N}^+$. Hence 1 is an upper bound for S. Now, let $\varepsilon > 0$ and consider $1 - \varepsilon$. We have that $1 - \varepsilon$ is no longer an upper bound for the set S since $1 \in S$ and $1 > 1 - \varepsilon$. Hence $1 = \sup(S)$. □

b) $0 = \inf(S)$.

Proof. To show 0 is the greatest lower bound, we will show that 0 is first a lower bound, then show $0+\varepsilon$ is no longer a lower bound for any $\varepsilon > 0$. It is clear that $0 < \frac{1}{n}$ for any $n \in \mathbb{N}^+$. Now let $\varepsilon > 0$ and consider $0 + \varepsilon = \varepsilon$. By the L.U.B 2 lemma in the book we have that for all $\varepsilon > 0$ there exists an $n \in \mathbb{N}^+$ such that $\frac{1}{n} < \varepsilon$. Hence, $\varepsilon > 0$ cannot be a lower bound for S. □

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3. Show if $A \subset B$ are **non-empty** subsets of $\mathbb{R}$ where B is bounded above, then A and B have least upper bounds and
$$\sup(A) \le \sup(B).$$
Find an example where $\sup(A) = \sup(B)$.

Note: Yet another small mistake was made in this homework assignment. It is necessary to assume that A and B non-empty. The result is not true if both A and B are both the empty set. The original has been changed to reflect this.

Solution: There are many examples where $A \subset B$ yet $\sup(A) = \sup(B)$. A couple of examples include:

a) $A = (0,1)$ and $B = [0,1]$

b) $A = \mathbb{Q} \cap (0,1)$ and $B = (0,1)$.

We will now prove the main result.

Proof. Suppose $A \subseteq B$ are subsets of $\mathbb{R}$ where B is bounded above. Since B is bounded above, by the least upper bound property of $\mathbb{R}$ we have that B has a least upper bound, call it $\beta = \sup(B)$. Moreover, since $A \subseteq B$ we have that A is also bounded above and by the least upper bound property A has a least upper bound, call it $\alpha = \sup(A)$. We must now show that $\alpha \le \beta$. However, since β is the an upper bound of B it is also an upper bound of A. Hence $\beta \ge a$ for all $a \in A$. Since α is the *least* upper bound of A we have by default, $\alpha \le \beta$. □

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4. Let A and B be two non-empty subsets of $\mathbb{R}$ which are bounded below. Show
$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

Solution: The proof of this result should be similar to an example we did in class.

Proof. By the hypothesis we have that A and B both have infimums. For notational convenience we let $\beta = \min\{\inf(A), \inf(B)\}$. Since A and B are non-empty we have that $A \cup B$ is non-empty. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \ge \inf(A) \ge \beta$. Likewise if $x \in B$ we have that $x \ge \beta$. Hence β is a lower bound for $A \cup B$ and thus $A \cup B$ has a greatest lower bound. Let $\gamma = \inf(A \cup B)$. By definition, since β is a lower bound we have $\beta \le \gamma$. Moreover, since $A \subseteq A \cup B$ we have that $\inf(A) \ge \gamma$ because γ is a lower bound for A and $\inf(A)$ is the *greatest* lower bound for A. Likewise $\inf(B) \ge \gamma$. Hence $\gamma \le \beta$ and thus $\beta = \gamma$. □

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5. Suppose that A and B are non-empty subsets of $\mathbb{R}$ that are bounded above. Let

$$A + B = \{a + b \ | a \in A, b \in B\}.$$

Show $A + B$ has a supremum and that $\sup(A + B) = \sup(A) + \sup(B)$.

Solution:

Proof. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. We have that $\alpha \geq a$ for all $a \in A$ and $\beta \geq b$ for all $b \in B$. Hence $\alpha + \beta \geq a + b$ for all $a + b \in A + B$ and $A + B$ is bounded above. Let $\gamma = \sup(A + B)$. We have that $\alpha + \beta$ is an upper bound, we now show that it is the least upper bound. Let $\varepsilon > 0$ and consider $(\alpha + \beta) - \varepsilon = (\alpha - \frac{\varepsilon}{2}) + (\beta - \frac{\varepsilon}{2})$. Since $\alpha - \frac{\varepsilon}{2} < \alpha$ there exists an $\hat{a} \in A$ such that $\hat{a} > \alpha - \frac{\varepsilon}{2}$. Likewise there exists a $\hat{b} \in B$ such that $\hat{b} > \beta - \frac{\varepsilon}{2}$. Hence,

$$\hat{a} + \hat{b} > (\alpha - \frac{\varepsilon}{2}) + (\beta - \frac{\varepsilon}{2}) = (\alpha + \beta) - \varepsilon.$$

Thus $(\alpha + \beta) - \varepsilon$ is not an upper bound for $A + B$ and we have $\alpha + \beta = \sup(A + B)$. □