MATH 3150
Homework 2 Solution

1. Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Define $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by
$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b).$$
Prove $(X \times Y, d)$ is a metric space.

Solution: We will show that $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ is a metric on $(X \times Y)$.

Let $(x, y), (a, b), (c, d) \in X \times Y$.

1) $d((x, y), (a, b)) \geq 0$

Proof. By definition,
$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b) \geq 0$$
since d_X and d_Y are both metrics. $\square$

2) $d((x, y), (a, b)) = 0$ if and only if $(x, y) = (a, b)$.

Proof. We note that $(x, y) = (a, b)$ if and only if $x = a$ and $y = b$. Since d_X and d_Y are both metrics we have that $d_X(x, a) = 0$ and $d_Y(y, b) = 0$ if and only if $x = a$ and $y = b$. Hence,
$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b) = 0$$
if and only if $(x, y) = (a, b)$. $\square$

3) $d((x, y), (a, b)) = d((a, b), (x, y))$

Proof. By definition,
$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b) = d_X(a, x) + d_Y(b, y) = d((a, b), (x, y))$$
since d_X and d_Y are both metrics. $\square$

4) $d((x, y), (a, b)) \leq d((x, y), (c, d)) + d((c, d), (a, b))$

Proof. By definition,
$$\begin{aligned} d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b) &\leq d_X(x, c) + d_X(c, a) + d_Y(y, d) + d_Y(d, b) \\ &= d((x, y), (c, d)) + d((c, d), (a, b)) \end{aligned}$$
since d_X and d_Y are both metrics.

$\square$

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2. Let X be a set with the following metric:

$$\rho(x,x) = 0$$

$$\rho(x,y) = 1, \quad x \neq y$$

Show that in (X, ρ) every subset is open.

Solution: We will show the result directly.

Proof. Suppose that S is a subset of X and let $x \in S$. We will show that there is a neighborhood around s contained in S. Notice that

$$N_{1/2}(s) = \left\{ x \in X \,\middle|\, \rho(x,s) < \frac{1}{2} \right\} = \{s\} \subset S$$

by the definition of the above metric ρ. □

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3. Let $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. Show that this sequence converges.

Hint: Show that this sequence is bounded above by 2 and increasing via induction.

Solution: We will prove by induction that the sequence is bounded above and increasing via induction then leverage these results to prove the main result.

Lemma 0.1. *The sequence (a_n) is increasing.*

Proof. We proceed by induction.

Base case: $a_1 \leq a_2$.

We simply notice that $a_1 = \sqrt{2} \leq \sqrt{2\sqrt{2}} = a_2$.

Induction Hypothesis: Assume that $a_n \leq a_{n+1}$.

We now show that $a_{n+1} \leq a_{n+2}$ given the induction hypothesis. Notice that,

$$a_{n+2} = \sqrt{2 \cdot a_{n+1}} \geq \sqrt{2 \cdot a_n} = a_{n+1}.$$

Hence the result is proven. □

Lemma 0.2. *The sequence (a_n) is bounded above by 2.*

Proof. We proceed by induction.

Base case: $a_1 \leq 2$.

We simply notice that $\sqrt{2} < 2$.

Induction Hypothesis: Assume that $a_n \leq 2$.

We now show that $a_{n+1} \leq 2$ given the induction hypothesis. Notice that,

$$a_{n+1} = \sqrt{2 \cdot a_n} \leq \sqrt{2 \cdot 2} = \sqrt{4} = 2.$$

Hence the result is proven. $\square$

Proof. By the above two lemmas we have that the sequence (a_n) is increasing and bounded above. By the monotone convergence theorem we have that the sequence converges. $\square$

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4. Find the limits and show by arguing directly from the definitions that the following sequences converge.

a) $a_n = \dfrac{2n-3}{n+5}$, $n \geq 0$.

b) $b_n = \dfrac{n+5}{n^2-n-1}$, $n \geq 2$.

Solution:

a) $a_n = \dfrac{2n-3}{n+5}$, $n \geq 0$.

Proof. We will show that $(a_n) \to 2$. We note that,

$$\left|\frac{2n-3}{n+5} - 2\right| = \left|\frac{2n-3-2(n+5)}{n+5}\right| = \left|\frac{-13}{n+5}\right| = \left|\frac{13}{n+5}\right|.$$

Let $\varepsilon > 0$ be given. Choose an $N \in \mathbb{N}$ such that

$$\left|\frac{13}{N+5}\right| < \varepsilon.$$

This is possible since $\{n+5 \mid n \in \mathbb{N}\}$ is an unbounded set of positive numbers. If $n \geq N$, then

$$\left|\frac{13}{n+5}\right| \leq \left|\frac{13}{N+5}\right| < \varepsilon.$$

Thus we have proven the intended limit. $\square$

b) $b_n = \dfrac{n+5}{n^2-n-1}$, $n \geq 2$.

Proof. We will show that $(b_n) \to 0$. We note that,

$$\begin{aligned}\left|\frac{n+5}{n^2-n-1} - 0\right| = \left|\frac{n+5}{n^2-n-1}\right| &\leq \left|\frac{n}{n^2-n-1}\right| + \left|\frac{5}{n^2-n-1}\right| \\ &= \left|\frac{1}{n-\left(1+\frac{1}{n}\right)}\right| + \left|\frac{5}{\left(n-\frac{1}{2}\right)^2 - \frac{5}{4}}\right| \\ &\leq \left|\frac{1}{n-3}\right| + \left|\frac{5}{n^2-\frac{5}{4}}\right|.\end{aligned}$$

There exists an $N_1 \in \mathbb{N}$ such that $\left|\frac{1}{N_1-3}\right| < \varepsilon/2$ and an $N_2 \in \mathbb{N}$ such that $\left|\frac{5}{N_2^2-\frac{5}{4}}\right| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$\left|\frac{1}{n-3}\right| + \left|\frac{5}{n^2-\frac{5}{4}}\right| \leq \left|\frac{1}{N-3}\right| + \left|\frac{5}{N^2-\frac{5}{4}}\right| < \varepsilon.$$

Thus we have proven the intended limit. □

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5. Suppose (a_n), (b_n) and (c_n) are sequences of real numbers. Show if $a_n \leq b_n \leq c_n$ for all n and both (a_n) and (c_n) converge to L then (b_n) converges to L.

Solution: We will prove the result directly.

Proof. We must show that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|b_n - L| < \varepsilon$. Let $\varepsilon > 0$ be given. Since $(a_n) \to L$ and $(c_n) \to L$ we have that there exists an N_a and N_c in $\mathbb{N}$ such that if $n \geq \max\{N_a, N_c\}$ we have $|a_n - L| < \varepsilon$ and $|c_n - L| < \varepsilon$. Note that

$$-\varepsilon < a_n - L \leq b_n - L \leq c_n - L < \varepsilon \quad \text{ if } n \geq \max\{N_a, N_c\}.$$

More specifically, if $n \geq \max\{N_a, N_c\}$ we have that $|b_n - L| < \varepsilon$. □

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6. Prove that a set is closed if and only if S contains all its limit points. As a reminder:

Definition 0.3. Let S be a subset of a metric space X. A point $y \in X$ is a limit point of S if and only if for every $\varepsilon > 0$ there exists a point $s \in S$ such that $s \neq y$ and $d(s, y) < \varepsilon$ (i.e. $N_\varepsilon(y) \cap (S \setminus \{y\}) \neq \emptyset$).

Solution: We showed in class that a set $S \subset X$ is closed if and only if every sequence from S which converges in X actually converges in S. We will use this proposition in our proof.

Proof. Suppose S is a closed set and $y \in X$ is a limit point of the set S. We will show there is a sequence from S which converges to y. Since S is closed the point y is then actually in S. Given $y \in X$ is a limit point of S, then for all $\varepsilon > 0$ there exists a point $s \in S$ such that $s \neq y$ and $d(s, y) < \varepsilon$. For each $n \in \mathbb{N}^+$, let $\varepsilon_n = \frac{1}{n}$. For each ε_n there exists a point $y \neq s_n \in S$ such that $d(s_n, y) < \varepsilon_n$ i.e. we have that $d(s_n, y) < \frac{1}{n}$. We have that the sequence s_n converges to y. Hence $y \in S$ since S is closed. Therefore, the set S contains all its limit points.

Conversely, suppose S contains all its limit points. We must show that S is closed. Let (s_n) be a sequence from S which converges to a point $s \in X$. We will show that s is actually in S. If $s_n = s$ for any n, then since (s_n) is a sequence from S our result follows. So suppose that $s_n \neq s$ for all n. Let $\varepsilon > 0$ be given. Since (s_n) converges to s there exists an $N_0 \in \mathbb{N}$ such that if $n \geq N_0$ then $d(s_n, s) < \varepsilon$ i.e. $N_\varepsilon(s) \cap (S \cap \{s\}) \neq \emptyset$. Hence $s \in S$ since s is a limit point. Therefore (s_n) converges to a point in S.

$\square$