MATH 3150 Your name
Homework 2 September 14, 2017

1. Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Define $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by
$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b).$$
Prove $(X \times Y, d)$ is a metric space.

2. Let X be a set with the following metric:
$$\rho(x, x) = 0$$
$$\rho(x, y) = 1, \quad x \neq y$$
Show that in (X, ρ) every subset is open.

3. Let $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. Show that this sequence converges.

   Hint: Show that this sequence is bounded above by 2 and increasing via induction.

4. Find the limits and show by arguing directly from the definitions that the following sequences converge.

   a) $a_n = \dfrac{2n - 3}{n + 5}$, $n \geq 0$.

   b) $b_n = \dfrac{n + 5}{n^2 - n - 1}$, $n \geq 2$.

5. Suppose (a_n), (b_n) and (c_n) are sequences of real numbers. Show if $a_n \leq b_n \leq c_n$ for all n and both (a_n) and (c_n) converge to L then (b_n) converges to L.

6. Prove that a set is closed if and only if S contains all its limit points. As a reminder:

   Definition 0.1. Let S be a subset of a metric space X. A point $y \in X$ is a limit point of S if and only if for every $\varepsilon > 0$ there exists a point $s \in S$ such that $s \neq y$ and $d(s, y) < \varepsilon$ (i.e. $N_\varepsilon(y) \cap (S \setminus \{y\}) \neq \emptyset$).