MATH 3210
Exam 1

1. Let $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ be the operator defined by

$$\begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} \mapsto \begin{bmatrix} X_0 \\ \vdots \\ X_{N-1} \end{bmatrix}$$

where

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N}.$$

This is the *discrete Fourier transform.* Define the map $\mathcal{D} : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\begin{bmatrix} X_0 \\ \vdots \\ X_{N-1} \end{bmatrix} \mapsto \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

where

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}.$$

Construct the matrices with respect to the standard basis (for both the domain and codomain) for both $\mathcal{D}$ and $\mathcal{F}$ on $\mathbb{C}^4$ and use these matrices to show $\mathcal{F}$ is invertible and its inverse is $\mathcal{D}$.

Hint: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Solution: Before we dive into a solution let's see where the discrete Fourier transform comes from. Let $C^1(\mathbb{T})$ denote the set of complex valued (or real valued) functions with domain $[0, 1]$ which are continuously differentiable and periodic, i.e.

$$C^1(\mathbb{T}) = \{x : [0,1] \to \mathbb{C} | x \text{ is continuously differentiable }, x(0) = x(1)\}$$

For a function in this space it is possible to represent it with a Fourier series, i.e.

$$x(t) = \sum_{n \in \mathbb{Z}} X[n] e^{i2\pi nt}$$

where

$$X[n] = \int_0^1 x(t) e^{-i2\pi nt} dt \tag{1}$$

Equation 1 is called the *Fourier transform* of the function $x(t)$. The Fourier transform is interpreted as transforming a function over the *time domain* into a function over the *frequency domain*. Some applications include signal analysis and solving differential equations. The formula

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N}$$

can be considered the Riemann sum *approximation* to the Fourier transform of $x(t)$ where the function $x(t)$ was sampled at a finite number of evenly spaced points. To find out more about the Fourier transform and the discrete Fourier transform consider looking here:

https://en.wikipedia.org/wiki/Fourier_transform

Now, onto the solution of problem 1. To construct the matrix of a linear transformation we apply the transformation to each basis element. For clarity, our basis is the following:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Below we expand out our formula for the entries of our vector.

$$X_0 = x_0 e^{\frac{-i2\pi\cdot 0\cdot 0}{4}} + x_1 e^{\frac{-i2\pi\cdot 0\cdot 1}{4}} + x_2 e^{\frac{-i2\pi\cdot 0\cdot 2}{4}} + x_3 e^{\frac{-i2\pi\cdot 0\cdot 3}{4}}$$

$$X_1 = x_0 e^{\frac{-i2\pi\cdot 1\cdot 0}{4}} + x_1 e^{\frac{-i2\pi\cdot 1\cdot 1}{4}} + x_2 e^{\frac{-i2\pi\cdot 1\cdot 2}{4}} + x_3 e^{\frac{-i2\pi\cdot 1\cdot 3}{4}}$$

$$X_2 = x_0 e^{\frac{-i2\pi\cdot 2\cdot 0}{4}} + x_1 e^{\frac{-i2\pi\cdot 2\cdot 1}{4}} + x_2 e^{\frac{-i2\pi\cdot 2\cdot 2}{4}} + x_3 e^{\frac{-i2\pi\cdot 2\cdot 3}{4}}$$

$$X_3 = x_0 e^{\frac{-i2\pi\cdot 3\cdot 0}{4}} + x_1 e^{\frac{-i2\pi\cdot 3\cdot 1}{4}} + x_2 e^{\frac{-i2\pi\cdot 3\cdot 2}{4}} + x_3 e^{\frac{-i2\pi\cdot 3\cdot 3}{4}}$$

Upon inspection, we see that the matrix for the discrete Fourier transform is given by:

$$[\mathcal{F}] = \begin{bmatrix} e^{\frac{-i2\pi\cdot 0\cdot 0}{4}} & e^{\frac{-i2\pi\cdot 0\cdot 1}{4}} & e^{\frac{-i2\pi\cdot 0\cdot 2}{4}} & e^{\frac{-i2\pi\cdot 0\cdot 3}{4}} \\ e^{\frac{-i2\pi\cdot 1\cdot 0}{4}} & e^{\frac{-i2\pi\cdot 1\cdot 1}{4}} & e^{\frac{-i2\pi\cdot 1\cdot 2}{4}} & e^{\frac{-i2\pi\cdot 1\cdot 3}{4}} \\ e^{\frac{-i2\pi\cdot 2\cdot 0}{4}} & e^{\frac{-i2\pi\cdot 2\cdot 1}{4}} & e^{\frac{-i2\pi\cdot 2\cdot 2}{4}} & e^{\frac{-i2\pi\cdot 2\cdot 3}{4}} \\ e^{\frac{-i2\pi\cdot 3\cdot 0}{4}} & e^{\frac{-i2\pi\cdot 3\cdot 1}{4}} & e^{\frac{-i2\pi\cdot 3\cdot 2}{4}} & e^{\frac{-i2\pi\cdot 3\cdot 3}{4}} \end{bmatrix}$$

Applying Euler's Formula gives us the following:

$$[\mathcal{F}] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

A similar process for $\mathcal{D}$ gives us

$$[\mathcal{D}] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

Since a linear transformation is invertible if and only if its matrix is invertible, a quick matrix multiplication shows,

$$[\mathcal{F}][\mathcal{D}] = \mathrm{Id}$$

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Hence $\mathcal{F}$ is an invertible transformation and $\mathcal{D}$ is its inverse.

2. Let $\mathcal{A}(\mathbb{R})$ be the space of "formal" power-series over the reals i.e.

$$\mathcal{A}(\mathbb{R}) = \left\{ f(x) = \sum_{n=0}^{\infty} a_n x^n \;\middle|\; a_i \in \mathbb{R} \right\}$$

with the usual operations of addition and scalar multiplication on powerseries. Let $\frac{d}{dx} : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ be the linear map of "differentiation", i.e.

$$\frac{d}{dx}(f(x)) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Let $\mathcal{A}_{n.c}(\mathbb{R})$ be the space of formal power series without a constant term, i.e.

$$\mathcal{A}_{n.c}(\mathbb{R}) = \left\{ f(x) = \sum_{n=1}^{\infty} a_n x^n \;\middle|\; a_i \in \mathbb{R} \right\}$$

Construct an explicit isomorphism $T : \mathcal{A}_{n.c}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})/\mathrm{null}(\frac{d}{dx})$.

Solution: As noted in class a possible candidate for our isomorphism is the following:

$$T : \sum_{n=1}^{\infty} a_n x^n \mapsto \sum_{n=1}^{\infty} a_n x^n + \text{null}\left(\frac{d}{dx}\right).$$

For notational convenience, let $[\sum_{n=0}^{\infty} b_n x^n]$ denote the element $\sum_{n=0}^{\infty} b_n x^n + \text{null}\left(\frac{d}{dx}\right)$ in the quotient space $A(\mathbb{R})/\text{null}\left(\frac{d}{dx}\right)$. Note that

$$\left[\sum_{n=0}^{\infty} b_n x^n\right] = \left[\sum_{n=0}^{\infty} c_n x^n\right]$$

if and only if there exits a constant $k \in \mathbb{R}$ such that

$$k = \sum_{n=0}^{\infty} b_n x^n - \sum_{n=0}^{\infty} c_n x^n.$$

Under this notation the map T looks rather natural since

$$T : \sum_{n=1}^{\infty} a_n x^n \mapsto \left[\sum_{n=1}^{\infty} a_n x^n\right] = \left[\sum_{n=0}^{\infty} a_n x^n\right]$$

where a_0 is any constant. We will now show that T is well-defined, linear, onto, and one to one.

T **is well-defined:**

Proof. This proof is more or less obvious, since if $\sum_{n=1}^{\infty} b_n x^n = \sum_{n=1}^{\infty} c_n x^n$ then $b_n = c_n$ for all $n \geq 1$. Hence, $0 = \sum_{n=1}^{\infty} b_n x^n - \sum_{n=1}^{\infty} c_n x^n$ and $[\sum_{n=1}^{\infty} b_n x^n] = [\sum_{n=1}^{\infty} c_n x^n]$. □

T **is linear:**

Proof. Again, this proof is more or less obvious. By the definition of addition in the quotient space we have that for all $\lambda \in \mathbb{R}$.

$$\lambda \sum_{n=1}^{\infty} b_n x^n + \sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} (\lambda b_n + c_n) x^n \mapsto \left[\sum_{n=1}^{\infty} (\lambda b_n + c_n) x^n\right] = \left[\lambda \sum_{n=1}^{\infty} b_n x^n + \sum_{n=1}^{\infty} c_n x^n\right]$$

and

$$\left[\lambda \sum_{n=1}^{\infty} b_n x^n + \sum_{n=1}^{\infty} c_n x^n\right] = \lambda \left[\sum_{n=1}^{\infty} b_n x^n\right] + \left[\sum_{n=1}^{\infty} c_n x^n\right].$$

□

T **is onto:**

Proof. Let $[\sum_{n=0}^{\infty} a_n x^n] \in A(\mathbb{R})/\text{null}\left(\frac{d}{dx}\right)$. It is clear that $\sum_{n=1}^{\infty} a_n x^n$ maps to $[\sum_{n=0}^{\infty} a_n x^n]$ under T. □

T **is one to one:**

Proof. Suppose $\sum_{n=1}^{\infty} b_n x^n \neq \sum_{n=1}^{\infty} c_n x^n$. There exists an $n \geq 1$ such that $b_n \neq c_n$. Hence there does not exist a constant $k \in \mathbb{R}$ such that $k = \sum_{n=1}^{\infty} b_n x^n - \sum_{n=1}^{\infty} c_n x^n$. Thus, $[\sum_{n=0}^{\infty} b_n x^n] \neq [\sum_{n=0}^{\infty} c_n x^n]$. □

3. Determine the dimension of $U = \left\{[a_1, \ldots, a_n]^\top \middle| \sum_{i=1}^{n} a_i = 0\right\}$ as a subspace of $\mathbb{R}^n$.

Hint: Consider the linear map $S : \mathbb{R}^n \to \mathbb{R}$ given by $S([a_1, \ldots, a_n]^\top) = \sum_{i=1}^{n} a_i$.

Solution: We will prove the result directly using rank-nullity.

Proof. Let S be the linear map $S : \mathbb{R}^n \to \mathbb{R}$ given by $S([a_1, \ldots, a_n]^\top) = \sum_{i=1}^{n} a_i$. Notice that $U = \text{null}(S)$. By rank-nullity,

$$\dim(\mathbb{R}^n) = \dim(\text{ran}(S)) + \dim(\text{null}(S)).$$

Note that $1 \in \text{ran}(S)$ since $[1/n, \ldots, 1/n]^\top \mapsto 1$ under S. Hence $\dim(\text{ran}(S)) = 1$. Thus $\dim(U) = \dim(\text{null}(S)) = n - 1$ since $\dim(\mathbb{R}^n) = n$.

□

4. Let $P_n(x) = \{p(x) = a_n x^n + \ldots + a_1 x + a_0 \mid a_i \in \mathbb{R},\ p : [0, 1] \to \mathbb{R}\}$ be the space of polynomials of degree $\leq n$. Let $P_{\text{per}}(x)$ be the subspace of polynomials in $P_n(x)$ with periodic boundary conditions, i.e.

$$P_{\text{per}}(x) = \{p \in P_n(x) \ \mid \ p(0) = p(1)\}.$$

Determine the dimension of $P_{\text{per}}(x)$ as a subspace of $P_n(x)$.

Hint: Try to construct a basis for $P_{\text{per}}(x)$ as a subspace of $P_2(x)$ and then generalize the argument for an arbitrary n. Alternatively, reduce to the above problem.

Solution: We claim that if $P_{\text{per}}(x) \subseteq P_n(x)$ has dimension n. We will prove the result in two ways. One by constructing a basis and one by reducing to the above problem.

Solution 1: We will use the following lemma, the proof of which is easily seen from the fundamental theorem of algebra.

Lemma 1: Suppose $p(x)$ is a non-constant polynomial with real coefficients. If $p(x)$ has a zero at $x = c$ then $p(x) = (x-c)s(x)$ for some polynomial $s(x)$ (with real coefficients).

We can now prove our main result. We will do so by constructing a basis for $P_{\text{per}}(x)$.

Proof. Note that $P_n(x)$ is an $n+1$ dimensional vector space. We claim $P_{\text{per}}(x)$ is n dimensional and that

$$\{v_0(x) = 1, v_1(x) = x(x-1), v_2(x) = x^2(x-1), \ldots, v_{n-1}(x) = x^{n-1}(x-1)\}$$

is a basis for the space. Let $p(x) \in P_{\text{per}}(x)$ where $p(x)$ is a non-constant polynomial. By our lemma,

$$p(x) = x(x-1)s(x)$$

where $\deg(s(x)) \leq n-2$. We have that $s(x) = a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$, where each $a_i \in \mathbb{R}$ for each $0 \leq i \leq n-2$. Hence, if $p(x)$ is a non-constant polynomial then

$$p(x) = x(x-1)s(x) = a_{n-2}x^{n-1}(x-1) + \ldots + a_1 x^2(x-1) + a_0 x(x-1).$$

If $p(x)$ is a constant polynomial then $p(x) = a \cdot 1$ for some $a \in \mathbb{R}$. Hence, $\{v_0(x), v_1(x) \ldots, v_{n-1}(x)\}$ generates $P_{\text{per}}(x)$. Moreover $\{v_0(x), v_1(x) \ldots, v_{n-1}(x)\}$ is a linearly independent set since if

$$0 = c_0 v_0(x) + c_1 v_1(x) \ldots + c_{n-1} v_{n-1}(x)$$

is a n degree polynomial which is 0 for all $x \in [0,1]$ then $c_0 v_0(x) + c_1 v_1(x) \ldots + c_{n-1} v_{n-1}(x)$ must be the zero polynomial and $c_0 = c_1 = \ldots = c_{n-1}$. □

Solution 2: We will now give a solution by reduction to the above problem. Suppose $p(x) = a_n x^n + \ldots + a_1 x + a_0$ is a polynomial of degree n such that $p(0) = p(1)$. Thus,

$$a_0 = p(0) = p(1) = a_n + \ldots + a_1 + a_0$$

i.e.

$$a_1 + \ldots + a_n = 0.$$

Hence, $p(x) = a_n x^n + \ldots + a_1 x + a_0$ has the property that $p(0) = p(1)$ if and only if $a_1 + \ldots + a_n = 0$. Define a linear isomorphism $T : P_n(x) \to \mathbb{R}^{n+1}$ by

$$T(a_0 + a_1 x + \ldots + a_n x^n) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Let $U \subseteq \mathbb{R}^n$ be the subspace defined by

$$U = \left\{ [a_0, a_1, \ldots, a_n]^\top \middle| \sum_{i=1}^{n} a_i = 0 \right\}.$$

Note that $T(P_{\text{per}}(x)) = U$. This is now similar to the previous problem. Construct a linear operator $S : \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$S([a_0, a_1, \ldots, a_n]) = \sum_{i=1}^{n} a_i$$

and note $U = \text{null}(S)$. By rank-nullity

$$n + 1 = \dim(\text{ran}(S)) + \dim(\text{null}(S))$$

Hence U is n-dimensional and its isomorphic image, $P_{\text{per}}(x)$, is n-dimensional.