MATH 3210
Exam 1

1. Let $\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$ be the operator defined by

$$\begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} \mapsto \begin{bmatrix} X_0 \\ \vdots \\ X_{N-1} \end{bmatrix}$$

where

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i2\pi kn/N}.$$

This is the *discrete Fourier transform.* Define the map $\mathcal{D} : \mathbb{C}^N \to \mathbb{C}^N$ given by

$$\begin{bmatrix} X_0 \\ \vdots \\ X_{N-1} \end{bmatrix} \mapsto \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

where

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{i2\pi kn/N}.$$

Construct the matrices with respect to the standard basis (for both the domain and codomain) for both $\mathcal{D}$ and $\mathcal{F}$ on $\mathbb{C}^4$ and use these matrices to show $\mathcal{F}$ is invertible and its inverse is $\mathcal{D}$.

Hint: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

2. Let $\mathcal{A}(\mathbb{R})$ be the space of "formal" power-series over the reals i.e.

$$\mathcal{A}(\mathbb{R}) = \left\{ f(x) = \sum_{n=0}^{\infty} a_n x^n \,\middle|\, a_i \in \mathbb{R} \right\}$$

with the usual operations of addition and scalar multiplication on powerseries. Let $\frac{d}{dx} : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ be the linear map of "differentiation", i.e.

$$\frac{d}{dx}(f(x)) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Let $\mathcal{A}_{n.c}(\mathbb{R})$ be the space of formal power series without a constant term, i.e.

$$\mathcal{A}_{n.c}(\mathbb{R}) = \left\{ f(x) = \sum_{n=1}^{\infty} a_n x^n \,\middle|\, a_i \in \mathbb{R} \right\}$$

Construct an explicit isomorphism $T : \mathcal{A}_{n.c}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})/\text{null}(\frac{d}{dx})$.

3. Determine the dimension of $U = \left\{[a_1, \ldots, a_n]^\top \middle| \sum_{i=1}^n a_i = 0\right\}$ as a subspace of $\mathbb{R}^n$.

Hint: Consider the linear map $S : \mathbb{R}^n \to \mathbb{R}$ given by $S([a_1, \ldots, a_n]^\top) = \sum_{i=1}^n a_i$.

4. Let $P_n(x) = \{p(x) = a_n x^n + \ldots + a_1 x + a_0 \mid a_i \in \mathbb{R},\ p : [0,1] \to \mathbb{R}\}$ be the space of polynomials of degree $\leq n$. Let $P_{\text{per}}(x)$ be the subspace of polynomials in $P_n(x)$ with periodic boundary conditions, i.e.

$$P_{\text{per}}(x) = \{p \in P_n(x) \ \mid \ p(0) = p(1)\}.$$

Determine the dimension of $P_{\text{per}}(x)$ as a subspace of $P_n(x)$.

Hint: Try to construct a basis for $P_{\text{per}}(x)$ as a subspace of $P_2(x)$ and then generalize the argument for an arbitrary n. Alternatively, reduce to the above problem.