1. Let $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Note that $\mathbb{Q}(\sqrt{2})$ is field and more specifically it is known as an algebraic number field. The binary operations on $\mathbb{Q}(\sqrt{2})$ are the standard addition and multiplication of numbers. Verify for all $\alpha \neq 0$ in $\mathbb{Q}(\sqrt{2})$ that there exists a $\beta \in \mathbb{Q}(\sqrt{2})$ such that $\alpha \cdot \beta = 1$.

   Solution: Consider $\alpha = a + b\sqrt{2} \neq 0$, where $a, b \in \mathbb{Q}$. Let

   $$\beta = \frac{1}{\alpha} = \frac{1}{a + b\sqrt{2}} \cdot \left(\frac{a - b\sqrt{2}}{a - b\sqrt{2}}\right) = \left(\frac{a}{a^2 - 2b^2}\right) - \left(\frac{b}{a^2 - 2b^2}\right)\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

   Clearly,

   $$\alpha \cdot \beta = \frac{a^2 + ab\sqrt{2} - ab\sqrt{2} - 2b^2}{a^2 - 2b^2} = 1.$$

   Note that if $a + b\sqrt{2} \neq 0$ where $b \neq 0$ then $a - b\sqrt{2} \neq 0$ (otherwise this implies $\sqrt{2} = \frac{a}{b}$) and

   $$a^2 - 2b^2 = \left(a + b\sqrt{2}\right) \cdot \left(a - b\sqrt{2} \neq 0\right) \neq 0.$$

2. Is the space of non-negative functions on the interval $[0, 1]$ a vector space over the real numbers $\mathbb{R}$? Justify your answer with a proof.

   Solution: The "space" of non-negative functions on the interval $[0, 1]$ is not a vector space over the real numbers. Under the standard operations it is not closed under scalar multiplication.

   Proof. Suppose $f : [0, 1] \to \mathbb{R}$ is a non-negative function on the interval $[0, 1]$ and let $c < 0$. If $f(x) \geq 0$ for all $x \in [0, 1]$ we have that $cf(x) \leq 0$ for all $x \in [0, 1]$. □

3. Let $M_{2\times 2}$ be the set of 2×2 matrices with real entries, i.e.

   $$M_{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}.$$

   $M_{2\times 2}$ is a vector space over the reals with the operations

   $$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} \text{ with } k \in \mathbb{R}$$

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix}$$

   Identify the additive identity in $M_{2\times 2}$ and justify your answer with a proof.

   Solution: The additive identity element in $M_{2\times 2}$ is the matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We will show O has the property that for any matrix $A \in M_{2\times 2}$ we have $A + O = A$. By uniqueness of the identity element in a vector space we must have that O is the identity.

Proof. Let $A \in M_{2\times 2}$ then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathbb{R}$. By definition

$$\begin{aligned} A + O &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= A \end{aligned}$$

□

4. Are the positive real numbers a field? Justify your answer.

   Solution: The positive real numbers are not a field because there is no additive identity element. The real numbers $\mathbb{R}$ by contrast are a field and the additive element is 0.

5. Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove $a = 0$ or $v = 0$.

   Solution: We will suppose that $a \neq 0$ and show that $v = 0$.

   Proof. Suppose $a \neq 0$ and show that $v = 0$. If $a \neq 0$ then there exists a unique multiplicative inverse element in the field, call it a^{-1}. If

$$av = 0$$

then

$$a^{-1}(av) = a^{-1}0 = 0,$$

and thus

$$v = 0.$$

□