1. Prove that the intersection of every collection of subspaces of V is a subspace of V. The following definition maybe helpful.

Definition 0.1. Let Γ be an arbitrary indexing set (possibly infinite and possibly uncountable). A collection of subspaces indexed by Γ is $\{U_\gamma \mid \gamma \in \Gamma, U_\gamma \text{ is a subspace of } V\}$.

Solution: We will show the result by applying the subspaces test.

Proof. A vector $u \in V$ is in $\bigcap_{\gamma\in\Gamma} U_\gamma$ if and only if $u \in U_\gamma$ for every $\gamma \in \Gamma$. To prove that $\bigcap_{\gamma\in\Gamma} U_\gamma$ is a subspace we will show that $0 \in \bigcap_{\gamma\in\Gamma} U_\gamma$ and that $\bigcap_{\gamma\in\Gamma} U_\gamma$ is closed under addition and scalar multiplication. Since each U_γ is a subspace then $0 \in U_\gamma$ for all $\gamma \in \Gamma$. Hence $0 \in \bigcap_{\gamma\in\Gamma} U_\gamma$. Likewise, let x and y be arbitrary vectors in $\bigcap_{\gamma\in\Gamma} U_\gamma$. Then $x \in U_\gamma$ and $y \in U_\gamma$ for all $\gamma \in \Gamma$. Since each U_γ is a subspace we have $x + y \in U_\gamma$ for all $\gamma \in \Gamma$. Hence $x + y \in \bigcap_{\gamma\in\Gamma} U_\gamma$. Similarly, since each U_γ is a subspace we have that $\lambda x \in U_\gamma$ for each $\lambda \in \mathbb{F}$, $x \in U_\gamma$ and each $\gamma \in \Gamma$. Thus $\lambda x \in \bigcap_{\gamma\in\Gamma} U_\gamma$.

□

2. Prove that the real vector space of all continuous real-valued functions on $[0, 1]$ is infinite dimensional.

Solution: We use the fact that if U is a subspace of V and V is finite dimensional then $\dim(U) \le \dim(V)$.

Proof. Suppose for the sake of contradiction that $V = C([0, 1], \mathbb{R})$ is finite dimensional. We note that $P(x) = \{p : [0, 1] \to \mathbb{R} \mid p \text{ is a polynomial}\}$ is a subspace of V. Moreover, $P(x)$ is an infinite dimensional vector space. Since, if we suppose that $\{p_1, \dots, p_m\}$ is a basis for $P(x)$. Let $n = \max\{\deg(p_1), \dots, \deg(p_m)\}$. Thus, $q(x) = x^n \notin \operatorname{span}\{p_1, \dots, p_m\}$. Thus, $\{p_1, \dots, p_m\}$ does not span $P(x)$ and we have a contradiction. Additionally, since $P(x)$ is infinite dimensional and a subspace of $V = C([0, 1], \mathbb{R})$ we have that V cannot be finite dimensional otherwise we would contradict the above fact. □

3. This exercise will walk you through a basic scheme for polynomial interpolation.

Polynomial Interpolation:

Given data

x_1	x_2	$\cdots$	x_n
a_1	a_2	$\cdots$	a_n

We want to compute a *interpolating polynomial* p, i.e. a polynomial of degree at most $n - 1$ such that

$$p(x_i) = f_i$$

Suppose you have a basis for the space of polynomials of $\deg(p) \leq n-1$, $P_{n-1}(x)$, say $\{p_1, p_2, \ldots, p_n\}$. If our interpolating polynomial p exists then

$$p(x) = c_1 p_1(x) + c_2 p_2(x) + \ldots + c_n p_n(x)$$

If p interpolates the data, then

$$p(x_1) = c_1 p(x_1) + c_2 p(x_1) + \ldots + c_n p(x_1) = a_1$$

$$p(x_2) = c_1 p(x_2) + c_2 p(x_2) + \ldots + c_n p(x_2) = a_2$$

$$\vdots$$

$$p(x_n) = c_1 p(x_n) + c_2 p(x_n) + \ldots + c_n p(x_n) = a_n$$

Thus we have to solve the linear system:

$$\begin{pmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_n(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_n(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_n(x_n) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Solutions:

(a) Find the matrix corresponding to the data points

$x_1 = 0$	$x_2 = -1$	$x_3 = 1$
2	3	3

and using the basis $\{p_1(x) = 1,\ p_2(x) = x,\ p_3(x) = x^2\}$

Solution to (a): The matrix is given by

$$\begin{pmatrix} p_1(x_1) & p_2(x_1) & p_3(x_1) \\ p_1(x_2) & p_2(x_2) & p_3(x_2) \\ p_1(x_3) & p_2(x_3) & p_3(x_3) \end{pmatrix} = \begin{pmatrix} p_1(0) & p_2(0) & p_3(0) \\ p_1(-1) & p_2(-1) & p_3(-1) \\ p_1(1) & p_2(1) & p_3(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Additionally, the linear system problem we need to solve in order to solve this polynomial interpolation problem is the following:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

Which, via row reduction, has solution $c_1 = 2, c_2 = 0, c_3 = 1$. Thus, our interpolating polynomial is $2p_1(x) + 1p_3(x) = 2 + x^2$.

(b) A more convenient basis for this problem is the Lagrange basis $\{L_1(x), \ldots, L_n(x)\}$ where the i-th Lagrange polynomial is given by

$$L_i(x) = \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

b.1) Find the Lagrange polynomials for the above data. Show that

$$L_i(x_k) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}.$$

Solution to b.1: Given then data points $(x_1, a_1) = (0, 2), (x_2, a_2) = (-1, 3)$, and $(x_3, a_3) = (1, 3)$ by the above formula, the Lagrange polynomials are the following:

$$\begin{aligned} L_1(x) &= \frac{(x - x_2)}{(x_1 - x_2)} \cdot \frac{(x - x_3)}{(x_1 - x_3)} = \frac{(x+1)}{(0+1)} \cdot \frac{(x-1)}{(0-1)} = (-1)(x+1)(x-1) \\ L_2(x) &= \frac{(x - x_1)}{(x_2 - x_1)} \cdot \frac{(x - x_3)}{(x_2 - x_3)} = \frac{(x)}{(-1-0)} \cdot \frac{(x-1)}{(-1-1)} = \frac{1}{2}(x)(x-1) \\ L_3(x) &= \frac{(x - x_1)}{(x_3 - x_1)} \cdot \frac{(x - x_2)}{(x_3 - x_2)} = \frac{(x)}{(1-0)} \cdot \frac{(x+1)}{(1+1)} = \frac{1}{2}(x)(x+1) \end{aligned}$$

One can easily check that

$$\begin{array}{lll} L_1(x_1) = 1, & L_2(x_1) = 0, & L_3(x_1) = 0 \\ L_1(x_2) = 0, & L_2(x_2) = 1, & L_3(x_2) = 0 \\ L_1(x_3) = 0, & L_2(x_3) = 0, & L_3(x_3) = 1 \end{array} \tag{1}$$

b.2) Use the above fact to show that the Lagrange polynomials are indeed a basis for $P_2(x)$.

Solution to b.2: We will show that the Lagrange polynomials are linearly independent. Since $\dim(P_2(x)) = 3$ this is enough to show that $\{L_1(x), L_2(x), L_3(x)\}$ is a basis.

Proof. Suppose that

$$q(x) = c_1 L_1(x) + c_2 L_2(x) + c_3 L_3(x) = 0.$$

We will show that $c_1 = c_2 = c_3 = 0$. By (1) we have that, $q(x_1) = c_1$, $q(x_2) = c_2$ and $q(x_3) = c_3$. However, $q(x) = 0$ for all x, thus we have that $c_1 = c_2 = c_3 = 0$. □

b.3) Compute the corresponding matrix to the above data and using the Lagrange polynomials as a basis.

Solution to b.3: Again by (1) we have that the corresponding matrix to the Lagrange polynomials is

$$\begin{pmatrix} L_1(x_1) & L_2(x_1) & L_3(x_1) \\ L_1(x_2) & L_2(x_2) & L_3(x_2) \\ L_1(x_3) & L_2(x_3) & L_3(x_3) \end{pmatrix} = \begin{pmatrix} L_1(0) & L_2(0) & L_3(0) \\ L_1(-1) & L_2(-1) & L_3(-1) \\ L_1(1) & L_2(1) & L_3(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Additionally, the linear system problem we need to solve in order to solve this polynomial interpolation problem is the following:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.$$

It is at this point we see the convenience of using the Lagrange polynomials as a basis. This linear system is *much* easier to solve (especially as the data set gets very large). Hence, our interpolating polynomial in terms of the Lagrange polynomial basis is given by

$$\begin{aligned} p(x) &= a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x) \\ &= -2(x+1)(x-1) + \frac{3}{2}(x)(x-1) + \frac{3}{2}(x)(x+1) \\ &= x^2 + 2 \end{aligned}$$