MATH 3210 Your name
Homework 2 January 29, 2018

1. Prove that the intersection of every collection of subspaces of V is a subspace of V. The following definition maybe helpful.

   Definition 0.1. Let Γ be an arbitrary indexing set (possibly infinite and possibly uncountable). A collection of subspaces indexed by Γ is $\{U_\gamma \mid \gamma \in \Gamma, U_\gamma \text{ is a subspace of } V\}$.

2. Prove that the real vector space of all continuous real-valued functions on $[0,1]$ is infinite dimensional.

3. This exercise will walk you through a basic scheme for polynomial interpolation.

   Polynomial Interpolation:

   Given data

x_1	x_2	$\cdots$	x_n
a_1	a_2	$\cdots$	a_n

   We want to compute a *interpolating polynomial* p, i.e. a polynomial of degree at most $n-1$ such that

$$p(x_i) = f_i$$

   Suppose you have a basis for the space of polynomials of $\deg(p) \leq n-1$, $P_{n-1}(x)$, say $\{p_1, p_2, \ldots, p_n\}$. If our interpolating polynomial p exists then

$$p(x) = c_1p_1(x) + c_2p_2(x) + \ldots + c_np_n(x)$$

   If p interpolates the data, then

$$p(x_1) = c_1p(x_1) + c_2p(x_1) + \ldots + c_np(x_1) = a_1$$

$$p(x_2) = c_1p(x_2) + c_2p(x_2) + \ldots + c_np(x_2) = a_2$$

$$\vdots$$

$$p(x_n) = c_1p(x_n) + c_2p(x_n) + \ldots + c_np(x_n) = a_n$$

   Thus we have to solve the linear system:

$$\begin{pmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_n(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_n(x_2) \\ \vdots & \vdots & \cdots & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_n(x_n) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Questions:

(a) Find the matrix corresponding to the data points

$x_1 = 0$	$x_2 = -1$	$x_3 = 1$
2	3	3

and using the basis $\{p_1(x) = 1,\, p_2(x) = x,\, p_3(x) = x^2\}$

(b) A more convenient basis for this problem is the Lagrange basis $\{L_1(x), \ldots, L_n(x)\}$ where the i-th Lagrange polynomial is given by

$$L_i(x) = \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

b.1) Find the Lagrange polynomials for the above data. Show that

$$L_i(x_k) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases} .$$

b.2) Use the above fact to show that the Lagrange polynomials are indeed a basis for $P_2(x)$.

b.3) Compute the corresponding matrix to the above data and using the Lagrange polynomials as a basis.