MATH 3210
Homework 3

For this homework assume all matrices are square and have entries from a field $\mathbb{F}$.

1. Show by induction that the determinant of an upper triangular matrix is the product of the diagonal entries.

   Solution: We will proceed by induction on the size of the matrix.

   Proof.
   Base Case: For a 1×1 matrix the result is obvious.

   Induction Hypothesis: Suppose that every upper triangular matrix of size $n \times n$ has determinant equal to the product of its diagonal, i.e. if A is $n \times n$ then

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

   Now suppose that A is a $(n+1) \times (n+1)$ upper triangular matrix. We compute the determinant of A by cofactor expansion along the first column. Hence,

$$\det(A) = a_{1,1} \det(\hat{A}_{1,1})$$

   We note that since A is upper triangular then $\hat{A}_{1,1}$ is upper triangular of size $n \times n$. By the induction hypothesis we have that

$$\det(A) = a_{1,1} \prod_{i=2}^{n+1} a_{i,i} = \prod_{i=1}^{n+1} a_{i,i}$$

   □

2. Call a matrix A nilpotent if $A^k = 0$ for some positive integer k. Show that every square nilpotent matrix has determinant zero.

   Solution: I will break the proof into two versions. In one version, the field should be either $\mathbb{C}$ or $\mathbb{R}$. In the second version, any field will work. I will accept either version as correct.

   Version 1:

   Proof. If $A^k = 0$ then $\det(A^k) = \det(0) = 0$. By the multiplicative property we have that

$$\det(A)^k = 0$$

   and hence

$$\det(A) = 0$$

   by taking k-th roots. □

Version 2:

Proof. Suppose that A is nilpotent but has a non-zero determinant. By the question above and the question below we have that A must be invertible. Let A^{-1} denote its inverse. Let k be the smallest positive integer such that $A^k = 0$.

$$I = A^{-1}A$$

$$A^{k-1} = IA^{k-1} = A^{-1}AA^{k-1} = A^{-1}A^k = A^{-1}0 = 0$$

Which is a contradiction on the condition we put on k. $\square$

3. Let A be an $n \times n$ matrix. Show from the definition of determinants that

$$\det(kA) = k^n \det(A)$$

for $k \in \mathbb{F}$.

Solution: We will proceed by induction on the size of the matrix.

Proof.
Base Case: For a 1×1 matrix the result is obvious.

Induction Hypothesis: Suppose for every matrix B of size $(n-1) \times (n-1)$ that $\det(kB) = k^{n-1} \det(B)$ for $k \in \mathbb{F}$.

Now suppose that A is an $n \times n$ matrix and $k \in \mathbb{F}$. We compute the determinant of kA by cofactor expansion along the first column. Thus,

$$\begin{aligned}
\det(kA) &= ka_{1,1} \det(k\hat{A}_{1,1}) + \ldots + ka_{n,1} \det(k\hat{A}_{n,1}) \\
&= k\left[a_{1,1} \det(k\hat{A}_{1,1}) + \ldots + a_{n,1} \det(k\hat{A}_{n,1})\right] \\
&= k \cdot k^{n-1}\left[a_{1,1} \det(\hat{A}_{1,1}) + \ldots + a_{n,1} \det(\hat{A}_{n,1})\right] \\
&= k^n\left[a_{1,1} \det(\hat{A}_{1,1}) + \ldots + a_{n,1} \det(\hat{A}_{n,1})\right] \\
&= k^n \det(A)
\end{aligned}$$

by our induction hypothesis since each $\hat{A}_{i,1}$ is an $(n-1) \times (n-1)$ matrix. $\square$

4. Let $T : V \to V$ where V is a finite dimensional vector space. Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{v_1', \ldots, v_n'\}$ be two ordered bases for V. Show there exists an invertible matrix P such that

$$[T]_\beta = P^{-1}[T]_\gamma P$$

Hint: Construct a linear transformation that switches out the basis and figure out how it relates to the matrix construction.

Solution: Before we prove the result we will recall how matrix multiplication represents the application of a linear transformation.

Review:

Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis for V and $\gamma = \{w_1, \ldots, w_m\}$ be an ordered basis for W. If $v \in V$ then $v = c_1v_1 + \ldots + c_nv_n$ for some unique choice of $c_1, \ldots, c_n$.

Define the linear transformation $\phi_\beta : V \to \mathbb{F}^n$ be defined by

$$\phi_\beta(v) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

For $T : V \to W$ and choices of bases β and γ (for V and W respectively) we can construct the matrix $[T]_\beta^\gamma$. If we denote $\phi_\beta(v)$ by $[v]_\beta$ (and likewise for ϕ_γ) we have the following relationship:

$$[T(v)]_\gamma = [T]_\beta^\gamma[v]_\beta. \tag{1}$$

For clarity, let $\hat{T}_\beta^\gamma : \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation given by multiplication by the matrix $[T]_\beta^\gamma$ i.e.

$$\hat{T}_\beta^\gamma(\vec{x}) = [T]_\beta^\gamma \cdot \vec{x}.$$

Note both ϕ_β and ϕ_γ are invertible. Hence,

$$T(v) = \phi_\gamma^{-1} \circ \hat{T}_\beta^\gamma \circ \phi_\beta(v)$$

by Equation (1). More specifically, applying T to a vector v is the same as applying the coordinate transformation ϕ_β, multiplying by $[T]_\beta^\gamma$, and then applying ϕ_γ^{-1}. Another way to say this is to say the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \\ \mathbb{F}^n \arrow[r, "\hat{T}_\beta^\gamma"] & \mathbb{F}^m \arrow[u, "\phi_\gamma^{-1}"'] \end{tikzcd}$$

We can now prove the result we want. Recall that $[T]_\beta$ is the simplified notation for $[T]_\beta^\beta$.

Proof. Let $T : V \to V$ where V is a finite dimensional vector space. Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{v_1', \ldots, v_n'\}$ be two ordered bases for V. From the above discussion we have the following combined diagram

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "\hat{T}_\gamma"] \arrow[d, "\phi_\gamma"'] & \mathbb{F}^n \\
V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[u, "\phi_\gamma^{-1}"'] \\
\mathbb{F}^n \arrow[r, "\hat{T}_\beta"] & \mathbb{F}^n \arrow[u, "\phi_\beta^{-1}"']
\end{tikzcd}$$

where $\hat{T}_\gamma$ and $\hat{T}_\beta$ are multiplication by $[T]_\gamma$ and $[T]_\beta$ respectively. Note,

$$\begin{aligned}\hat{T}_\gamma &= \phi_\gamma \circ T \circ \phi_\gamma^{-1} \\ &= \phi_\gamma \circ \phi_\beta^{-1} \circ \hat{T}_\beta \circ \phi_\beta \circ \phi_\gamma^{-1}.\end{aligned}$$

These linear transformations have matrices (with respect to the standard basis for $\mathbb{F}^n$). Hence,

$$[T]_\gamma = P^{-1}[T]_\beta P$$

where P is the matrix for the transformation $\phi_\beta \circ \phi_\gamma^{-1} : \mathbb{F}^n \to \mathbb{F}^n$. □

Note: A similar proof results in a more general result:

Proposition 0.1. *If β and β' are bases for V and γ and γ' are bases for W then*

$$[T]_{\beta'}^{\gamma'} = Q^{-1}[T]_\beta^\gamma P$$

for some invertible matrices Q and P.

5. Define the determinant of a linear transformation $T : V \to V$ by

$$\det(T) = \det([T]_\beta)$$

where $[T]_\beta$ is a matrix of T with respect to an ordered basis. Does this definition rely on the choice of basis ?

Solution: This definition does not depend on the choice of basis. If $A = P^{-1}BP$ then

$$\det(A) = \det(P^{-1}BP) = \det(P^{-1})\det(B) = \det(P) = \det(B)$$

since $\det(P^{-1}) = \frac{1}{\det(P)}$. Thus, $\det([T]_\gamma) = \det([T]_\beta)$ for any choices of bases γ and β.