MATH 3210
Homework 4 Solution

1. Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{ran}(S)$ is invariant under T

   Solution: We prove the result directly.

   Proof. If $v \in \operatorname{ran}(S)$ then $v = S(u)$ for some $u \in V$. We apply T to v. Hence we have that
$$T(v) = TS(u) = ST(u).$$
   Thus $T(v) = S(T(u))$ and is in the range of S. □

2. Let $V = (\mathbb{Z}/5\mathbb{Z})^3$. Define, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}/5\mathbb{Z}$ by
$$\langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 \qquad \text{for all } \vec{x}, \vec{y} \in V.$$
   Is $\langle \cdot, \cdot \rangle$ an inner product?

   Solution: This "inner-product" fails to have the property that $\langle v, v \rangle > 0$ for all non-zero v. As a counter example let $v = [2,\ 1,\ 0]^\top$ and note that $\langle v, v \rangle = 2^2 + 1^2 + 0^2 \equiv_5 0$.

3. Let $V = (\mathbb{Z}/5\mathbb{Z})^2$ and $T : V \to V$ be the transformation $T(\vec{x}) = A \cdot \vec{x}$ where A is given by
$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \in M_{2\times 2}(\mathbb{Z}/5\mathbb{Z}).$$
   Does T have eigenvalues and eigenvectors? If so, find them and determine if T has a diagonal matrix with respect to a basis of eigen-vectors.

   Solution: Since A is upper triangular we note that T has eigenvalues 2 and 4. Moreover, T does have eigenvectors. Namely,
$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$
   and
$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
   where the operations are taken modulo 5. The vector space $V = (\mathbb{Z}/5\mathbb{Z})^2$ has a basis of eigenvectors for T. Since
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq c \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$
   for all $c \in \mathbb{Z}/5\mathbb{Z}$. Thus, the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ is linearly independent and hence a basis for V (note, $\dim(V) = 2$).

4. In class we defined for a polynomial $p(x) = a_nx^n + \ldots + a_1x + a_0$ and an operator $T \in \mathcal{L}(V)$ the operator $p(T)$ as
$$p(T) = a_nT^n + \ldots + a_1T + a_0I \in \mathcal{L}(V).$$

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Let $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and define for a operator $T \in \mathcal{L}(V)$

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}.$$

[If you have taken analysis do not worry about convergence, the power series has an infinite radius of convergence.]

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Find a formula for A^n and prove it by induction.

(b) Find e^A.

Solution:

(a) We claim the following formula holds

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

Proof. We proceed by induction on n.

Base case: n=1

This case is obvious.

Induction Hypothesis: Suppose for $n = k - 1$ we have that

$$A^{k-1} = \begin{bmatrix} 1 & k-1 \\ 0 & 1 \end{bmatrix}.$$

We now calculate A^k. By the induction hypothesis

$$A^k = A^{k-1}A = \begin{bmatrix} 1 & k-1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

□

(b) Calculating directly,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} & \sum_{n=0}^{\infty} \frac{n}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{1}{n!} \end{bmatrix} = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}$$

5. The Fibonacci sequence $F_1, F_2, \ldots$ is defined by

$$F_1 = 1, F_2 = 1, \qquad \text{and} \qquad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x+y \end{bmatrix}.$$

(a) Show that $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$

(b) Find the eigenvalues of T.

(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

(d) Use the solution to part (c) to compute $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each positive integer n.

Solution:

(a) We will prove the result by induction on n.

Proof.
Base case: $n = 1$

Note that $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$.

Induction Hypothesis: Suppose for $n = k-1$ we have that

$$T^{k-1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_{k-1} \\ F_k \end{bmatrix}$$

Now we apply T to T^{k-1} and we have that

$$T^k\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} F_{k-1} \\ F_k \end{bmatrix}\right) = \begin{bmatrix} F_k \\ F_{k-1} + F_k \end{bmatrix} = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$$

by application of the recurrence relation and induction hypothesis. □

(b) The eigenvector equation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

is equivalent to the system

$$y = \lambda x \qquad \text{and} \qquad x + y = \lambda y.$$

By substitution we have that

$$x + \lambda x = \lambda^2 x.$$

We note that $x \neq 0$ since this would imply $y = 0$ by the system of equations and the zero vector is not a candidate for an eigenvector. Hence we can dived both sides by x and get that

$$\lambda^2 - \lambda - 1 = 0.$$

The only solutions to this equation are

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

(c) We find the eigenvectors corresponding to the above eigenvectors. Substituting $\lambda = \frac{1\pm\sqrt{5}}{2}$ into the above system and solving for x and y shows that the eigenvectors are

$$\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}. \quad \text{and} \quad \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

These vectors are clearly linearly independent and thus form a basis.

(d) Note that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

Hence,

$$T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{5}} T^n\left(\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}\right) - \frac{1}{\sqrt{5}} T^n\left(\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}\right).$$

By our eigenvalue relation ship we have that

$$T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

By part (a) we have

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$$

for each positive integer n.