MATH 3210 Your name
Homework 4 date

1. Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\text{ran}(S)$ is invariant under T

2. Let $V = (\mathbb{Z}/5\mathbb{Z})^3$. Define, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}/5\mathbb{Z}$ by
$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 \qquad \text{for all } \vec{x}, \vec{y} \in V.$$
Is $\langle \cdot, \cdot \rangle$ an inner product?

3. Let $V = (\mathbb{Z}/5\mathbb{Z})^2$ and $T : V \to V$ be the transformation $T(\vec{x}) = A \cdot \vec{x}$ where A is given by
$$A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \in M_{2\times 2}(\mathbb{Z}/5\mathbb{Z}).$$
Does T have eigenvalues and eigenvectors? If so, find them and determine if T has a diagonal matrix with respect to a basis of eigen-vectors.

4. In class we defined for a polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$ and an operator $T \in \mathcal{L}(V)$ the operator $p(T)$ as
$$p(T) = a_n T^n + \ldots + a_1 T + a_0 I \in \mathcal{L}(V).$$
Let $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and define for a operator $T \in \mathcal{L}(V)$
$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}.$$
[If you have taken analysis do not worry about convergence, the power series has an infinite radius of convergence.]

   Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   (a) Find a formula for A^n and prove it by induction.

   (b) Find e^A.

5. The Fibonacci sequence $F_1, F_2, \ldots$ is defined by
$$F_1 = 1, F_2 = 1, \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3$$
Define $T \in \mathcal{L}(\mathbb{R}^2)$ by
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x + y \end{bmatrix}.$$

   (a) Show that $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$

   (b) Find the eigenvalues of T.

   (c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

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(d) Use the solution to part (c) to compute $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each positive integer n.