

1. Let  $C_0^1[0, 1] = \{f : [0, 1] \rightarrow \mathbb{C} \mid f \text{ is continuously differentiable and } f(0) = 0 = f(1)\}$ , i.e. it is the vector-space of functions with a continuous derivative which are zero at the end points. Let

$$\langle f, g \rangle = \int_0^1 f(x)\bar{g}(x) dx$$

be an inner-product on this space. Define a map  $T : C_0^1[0, 1] \rightarrow C[0, 1]$  by  $T(f) = -i\frac{df}{dx}$ . Show that,

$$\langle Tf, g \rangle = \langle f, Tg \rangle.$$

**Hint:** Use integration by parts.

2. Show that a normal operator is self adjoint if and only if its eigenvalues are real.
3. Let  $U \in \mathcal{L}(V)$  be called a unitary operator if  $U^*U = UU^* = I$ .
- (a) Show that for all  $v \in V$  that  $\|v\| = \|Uv\|$ .
  - (b) Show that if  $\lambda$  is an eigenvalue of  $U$  then  $|\lambda| = 1$ .
  - (c) Show that if  $\{e_1, e_2, \dots, e_n\}$  is an orthonormal basis then  $\{Ue_1, Ue_2, \dots, Ue_n\}$  is an orthonormal basis.
  - (d) Show that if  $S$  is an operator such that if  $\{e_1, \dots, e_n\}$  is an orthonormal basis then  $\{Se_1, \dots, Se_n\}$  is an orthonormal basis then  $S$  is unitary.
4. Call a matrix  $U$  unitary if the operator  $S(x) = Ux$  is a unitary operator. Let  $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a normal operator given by  $T(x) = Ax$  where  $A$  is an  $n \times n$  matrix ( $A$  is the matrix for  $T$  with respect to the standard basis.) Show that there exists a unitary matrix  $U$  such that  $U^{-1}AU = D$  where  $D$  is a diagonal matrix.