MATH 3210 Your name
Homework 5 date

1. Let $C_0^1[0,1] = \{f : [0,1] \to \mathbb{C} \mid f \text{ is continuously differentiable and } f(0) = 0 = f(1)\}$, i.e. it is the vector-space of functions with a continuous derivative which are zero at the end points. Let

$$\langle f, g \rangle = \int_0^1 f(x)\bar{g}(x)\,dx$$

be an inner-product on this space. Define a map $T : C_0^1[0,1] \to C[0,1]$ by $T(f) = -i\frac{df}{dx}$. Show that,

$$\langle Tf, g \rangle = \langle f, Tg \rangle.$$

Hint: Use integration by parts.

2. Show that a normal operator is self adjoint if and only if its eigenvalues are real.

3. Let $U \in \mathcal{L}(V)$ be called a unitary operator if $U^*U = UU^* = I$.

   (a) Show that for all $v \in V$ that $\|v\| = \|Uv\|$.

   (b) Show that if λ is an eigenvalue of U then $|\lambda| = 1$.

   (c) Show that if $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis then $\{Ue_1, Ue_2, \ldots, Ue_n\}$ is an orthonormal basis.

   (d) Show that if S is an operator such that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis then $\{Se_1, \ldots, Se_n\}$ is an orthonormal basis then S is unitary.

4. Call a matrix U unitary if the operator $S(x) = Ux$ is a unitary operator. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a normal operator given by $T(x) = Ax$ where A is an $n \times n$ matrix (A is the matrix for T with respect to the standard basis.) Show that there exists a unitary matrix U such that $U^{-1}AU = D$ where D is a diagonal matrix.