Contents

1 Basic Notions

Definition 1.1. A binary operation on a set S is a function $f : S \times S \to S$.

Definition 1.2. A *field* is a set $\mathbb{F}$ together with two binary operations $+$, and $\cdot$ called addition and multiplication (respectively) such that

1. For all $a, b, c \in \mathbb{F}$ we have
$$a + (b + c) = (a + b) + c$$
and
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

2. For all $a, b \in \mathbb{F}$ we have
$$a + b = b + a$$
and
$$a \cdot b = b \cdot a.$$

3. There exists an element $0 \in \mathbb{F}$, called an additive identity, such that for all $a \in \mathbb{F}$ we have $a + 0 = a$.

4. There exists an element $1 \in \mathbb{F}$, called a multiplicative identity, such that for all $a \in \mathbb{F}$ we have $a \cdot 1 = a$.

5. For all $a \in \mathbb{F}$ there exists an element $b \in \mathbb{F}$, called an additive inverse, such that $a+b = 0$.

6. For all $a \in \mathbb{F}$ such that $a \neq 0$ there exists an element $c \in \mathbb{F}$, called a multiplicative inverse, such that $a \cdot c = 1$.

7. For all $a, b, c \in \mathbb{F}$
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Note: Fields have a unique additive and multiplicative identity denoted 0 and 1 respectively. Moreover, when the additive and multiplicative inverses exist they are unique.

Example 1.3. All of the following examples are with their standard operations.

1. $\mathbb{Q}$ (rational numbers)

2. $\mathbb{R}$ (real numbers)

3. $\mathbb{C}$ (complex numbers)

4. $\mathbb{Z}/p\mathbb{Z}$ for p prime (Integers modulo p)

Non example: $\mathbb{Z}$ is not a field, it lacks multiplicative inverses.

Definition 1.4. A *vector space* V over a field $\mathbb{F}$ is a set V with two operations called *vector addition* and *scalar multiplication* where vector addition is a function $+ : V \times V \to V$ and scalar multiplication is a function $\cdot : \mathbb{F} \times V \to V$ such that

1. For all $u, v \in V$ we have
$$u + v = v + u$$
2. For all $u, v, w \in V$ and for all $a, b \in \mathbb{F}$ we have
$$(u + v) + w = u + (v + w)$$
and
$$(ab) \cdot v = a \cdot (b \cdot v)$$
3. There exists a vector $0 \in V$, called an additive identity, such that for all $v \in V$ we have
$$v + 0 = v$$
4. For all $v \in V$ we have a vector $w \in V$, called an additive inverse, such that
$$v + w = 0$$
5. For all $v \in V$ we have
$$1 \cdot v = v$$
6. For all $a, b \in \mathbb{F}$ and for all $u, v \in V$ we have
$$a \cdot (u + v) = a \cdot u + a \cdot v$$

Example 1.5. All of the following examples are with their standard operations.

1. $\mathbb{F}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{F} \right\}$ where $\mathbb{F}$ is a field.
2. Polynomials with coefficients in a field $\mathbb{F}$.
3. Polynomials (with coefficients in a field $\mathbb{F}$) of degree $\leq n$

4. Continuous functions $f : X \to Y$, $C(X, Y)$, where X and Y are fields.

5. Functions from a field X into a field Y.

6. $\mathbb{F}^\infty = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{F}\}$.

Proposition 1.1. *Every vector space V has a unique additive identity. The unique additive identity is denoted 0.*

Proposition 1.2. *Every element $v \in V$ has a unique additive inverse. For all $v \in V$ its unique additive inverse is denoted $-v$.*

Proposition 1.3. *For all $v \in V$ we have $0 \cdot v = 0$.*

Proposition 1.4. *For all $a \in \mathbb{F}$ and $0 \in V$ we have $a \cdot 0 = 0$.*

Proposition 1.5. *For every $v \in V$ we have $(-1) \cdot v = -v$*

2 Basis for a Vector Space

Definition 2.1. A *linear combination* of a list of vectors $v_1, \ldots, v_m$ in V is a vector of the form

$$a_1 v_1 + \ldots + a_m v_m$$

where $a_1, \ldots, a_m \in \mathbb{F}$.

Definition 2.2. The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$ denoted by $\operatorname{span}\{v_1, \ldots, v_m\}$.

$$\operatorname{span}\{v_1, \ldots, v_m\} = \{a_1 v_1 + \ldots + a_m v_m \mid a_i \in \mathbb{F}\}$$

Definition 2.3. If V is a vector space and $V = \operatorname{span}\{v_1, \ldots, v_m\}$ then we say that $v_1, \ldots, v_m$ span V.

Definition 2.4. We say that a vectors space is *finite dimensional* if there exists a finite list of vectors $v_1, \ldots, v_m$ such that

$$\operatorname{span}\{v_1, \ldots v_m\} = V$$

Otherwise we say that V is *infinite dimensional.*

Definition 2.5. A list of vectors $v_1, \ldots, v_m$ in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$a_1 v_1 + \ldots + a_m v_m = 0$$

is $a_1 = a_2 = \ldots = a_m$. A list is called *linearly dependent* if it is not linearly independent.

Lemma 2.6. *Suppose that $v_1, \ldots, v_m$ is a linearly dependent list in V. There exists a $j \in \{1, \ldots, m\}$ such that*

1) $v_j \in \operatorname{span}\{v_1, \ldots v_{j-1}\}$

2) $\operatorname{span}\{v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots v_m\} = \operatorname{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_m\}$

Proposition 2.1. *In a finite dimensional vector space the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.*

Definition 2.7. A basis for a vector space V is a list of vectors $\{v_1, \ldots, v_n\}$ such that

1. $\{v_1, \ldots, v_n\}$ is linearly independent

2. $\operatorname{span}\{v_1, \ldots, v_n\} = V$.

Proposition 2.2. *A list of vectors $\{v_1, \ldots, v_n\}$ in V is a basis for V if and only if every vector $v \in V$ can be written* uniquely *in the form*

$$v = a_1 v_1 + \ldots + a_n v_n$$

for some $a_1, \ldots, a_n \in \mathbb{F}$.

Proposition 2.3. *Every spanning list of vectors in V can be reduced down to a basis.*

Proposition 2.4. *Every linearly independent list of vectors in V can be extended to a basis.*

Proposition 2.5. *Any two basis of a finite dimensional vector space V have the same length.*

Definition 2.8. The dimension of a finite dimensional vector space is the length of any basis of the vector space. Denoted $\dim(V)$.

Proposition 2.6. *Suppose V is finite dimensional. Every list of linearly independent vectors whose length is equal to the dimension of V is a basis.*

Proposition 2.7. *Suppose V is finite dimensional. Every spanning list vectors whose length is equal to the dimension of V is a basis.*

3 Subspaces

Definition 3.1. A subspace of a vector space V is a subset H such that H is a vector space under the same binary relations and field as V.

Proposition 3.1 (Subspace Test)**.** *A subset H is a subspace of V if and only if*

1. $0 \in H$.

2. *For all $u, v \in H$ we have $u + v \in H$*

3. *For all $u \in H$ and $a \in \mathbb{F}$ we have $au \in H$.*

Proposition 3.2. *If U is a subspace of a finite dimensional vector space V then* $\dim(U) \leq \dim(V)$. *Moreover,* $\dim(U) = \dim(V)$ *if and only if $V = U$.*

3.1 Direct Sums

Definition 3.2. Suppose $U_1, \ldots, U_m$ are subsets of V. The *sum* of $U_1, \ldots, U_m$ denoted $U_1 + \ldots + U_m$ is the set of all possible sums i.e.,

$$U_1 + \ldots + U_m = \{u_1 + \ldots + u_m \mid u_i \in U_i, i = 1, \ldots, m\}$$

Proposition 3.3. *If $U_1, \ldots, U_m$ are subspaces then so is $U_1 + \ldots + U_m$.*

Definition 3.3. Suppose $U_1, \ldots, U_m$ are subspaces of V. The sum $U_1 + \ldots + U_m$ is a *direct sum* if each element of $U_1 + \ldots + U_m$ can be written in only one way as a sum $u_1 + \ldots + u_m$ where $u_i \in U_i$, $i = 1, \ldots, m$. The direct sum is denoted $U_1 \oplus \ldots \oplus U_m$.

Proposition 3.4. *$U_1 + \ldots + U_m$ is a direct sum if and only if the only way to write* 0 *as a sum is by taking each u_i where $i = 1, \ldots, m$ to be* 0.

Proposition 3.5. *The sum of two subspaces U and W is a direct sum if and only if $U \cap W = \{0\}$.*

Proposition 3.6. *If V is a finite dimensional vector space and U is a subspace of V then there exists a W which is a subspace of V such that $V = U \oplus W$.*

3.2 Quotient Spaces

Definition 3.4. Let V be a vectors space and U a subspace. For every $v \in V$ define

$$v + U = \{v + u \mid u \in U\}$$

and

$$V/U = \{v + U \ \mid \ v \in V\}$$

Proposition 3.7. *Let V be a vectorspace, U a subspace and $v, w \in V$. The following are equivalent.*

(a) $v - w \in U$

(b) $v + U = w + U$

(c) $(v + U) \cap (w + U) \neq \emptyset$

Proposition 3.8. *Let V be a vectorspace, U a subspace, $\lambda \in \mathbb{F}$, and $v, w \in V$ The set V/U is a vector space with the following operations:*

$$(v + U) + (w + U) = (v + w) + U$$

$$\lambda(v + U) = (\lambda v) + U$$

Proposition 3.9. *Let V be a finite dimensional vector space and U be a subspace.*

$$\dim(V/U) = \dim(V) - \dim(U)$$

4 Linear Maps

Definition 4.1. A linear map from $V(\mathbb{F})$ to $W(\mathbb{F})$ is a function $T : V \to W$ such that

$$T(\lambda x + y) = \lambda T(x) + T(y)$$

for every $x, y \in V$ and $\lambda \in \mathbb{F}$. Denote the set of all linear maps from V to W as $\mathcal{L}(V, W)$.

Proposition 4.1. *Suppose $\{v_1, \ldots, v_n\}$ is a basis of V and $w_1, \ldots, w_n \in W$. There is a unique linear map $T : V \to W$ such that*

$$T(v_j) = w_j \qquad j = 1, \ldots, n.$$

Proposition 4.2. *If $T : V \to W$ is linear then*

$$T(0_V) = 0_W.$$

Definition 4.2. Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$ define

$$(S+T)(v) = S(v) + T(v) \text{ for all } v \in V$$

and

$$(\lambda T)(v) = \lambda T(v) \text{ for all } v \in V.$$

Proposition 4.3. *$\mathcal{L}(V, W)$ is a vector space with the above operations.*

4.1 One to One, Onto, Invertibility, and Isomorphisms

Definition 4.3. For $T \in \mathcal{L}(V, W)$ define the *null space* of T (or kernel of T) to be

$$\text{null}(T) = \{v \in V \mid Tv = 0\} \subseteq V$$

and define the *range* of T to be

$$\text{ran}(T) = \{Tv \mid v \in V\} \subseteq W.$$

Theorem 4.4. *For $T \in \mathcal{L}(V, W)$ both $\text{null}(T)$ and $\text{ran}(T)$ are subspaces. A linear transformation is injective if and only if*

$$\text{null}(T) = \{0\}.$$

A linear transformation is surjective if and only if $\text{ran}(T) = W$.

Definition 4.5. The *rank* of a linear transformation is the dimension of its range. The *nullity* of a transformation is the dimension of its null space.

Definition 4.6. A vector space isomorphism from V onto W is a bijective linear map $T : V \to W$. If there is a vector space isomorphism from V onto W we say V is isomorphic to W and write $V \cong W$.

Theorem 4.7. *Let V and W be two finite dimensional vector spaces. V is isomorphic to W if and only if $\dim(V) = \dim(W)$.*

Definition 4.8. A linear map $T : V \to W$ is *invertible* if there exists an map $S : W \to V$ such that

$$S \circ T = \text{Id}_V$$

$$T \circ S = \text{Id}_W$$

Theorem 4.9. *A map $T : V \to W$ is invertible if and only if its bijective.*

4.2 Fundamental Theorem of Linear Maps

Proposition 4.4. *Suppose $T \in \mathcal{L}(V, W)$ and define*

$$\tilde{T} : V/\operatorname{null}(T) \to W$$

by

$$\tilde{T}(v + \operatorname{null}(T)) = Tv.$$

The following hold:

a) $\tilde{T}$ *is linear*

b) $\tilde{T}$ *is injective*

c) $\operatorname{ran}(\tilde{T}) = \operatorname{ran}(T)$

d) $V/\operatorname{null}(T) \cong \operatorname{ran}(T)$*.*

Theorem 4.10 (Fundamental Theorem of Linear Maps/ Rank-Nullity)**.** *Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V, W)$. We have* $\operatorname{ran}(T)$ *is finite dimensional and*

$$\dim(V) = \dim(\operatorname{ran}(T)) + \dim(\operatorname{null}(T)).$$

Proposition 4.5. *Suppose V and W are finite dimensional vector spaces. Let $T \in \mathcal{L}(V, W)$.*

a) If $\dim(V) < \dim(W)$ *then T is not surjective.*

b) If $\dim(V) > \dim(W)$ *then T is not injective.*

Definition 4.11. A linear map $T \in \mathcal{L}(V, V)$ is called an *operator*.

Theorem 4.12. *Suppose T is an operator over a vector space V. If V is finite dimensional the following are equivalent:*

a) T is injective

b) T is surjective

c) T is bijective

4.3 The Matrix of a Linear Map and the Coordinate Transform

Definition 4.13. Let V be an n-dimensional vector space over the field $\mathbb{F}$. Let $\beta = \{v_1, \dots, v_n\}$ be an *ordered* basis for V. The coordinate transform $\varphi_\beta : V \to \mathbb{F}^n$ is defined by

$$v = a_1 v_1 + \ldots + a_n v_n \xmapsto{\varphi_\beta} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Denote the column vector $\varphi_\beta(v)$ by $[v]_\beta$.

Proposition 4.6. *The coordinate transform from V onto $\mathbb{F}^n$ is a vector space isomorphism.*

Definition 4.14. Suppose $T : V \to W$ is a linear maps with $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ as ordered basis for V and W respectively. Define the matrix of T, denoted by $[T]_\beta^\gamma$, by the following If

$$T(v_k) = a_{1,k} w_1 + \ldots + a_{m,k} w_k$$

then the k-th column of $[T]_\beta^\gamma$ is given by $\begin{bmatrix} a_{1,k} \\ \vdots \\ a_{m,k} \end{bmatrix}$.

Theorem 4.15. *The following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \\ \mathbb{F}^n \arrow[r, "{[T]_\beta^\gamma}"] & \mathbb{F}^m \arrow[u, "\phi_\gamma^{-1}"'] \end{tikzcd}$$

More specifically,

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma.$$

5 Determinants

Definition 5.1. Let $\hat{A}_{i,j}$ be the $(n-1) \times (n-1)$ matrix that results from A by removing the ith row and jth column and let $a_{i,j}$ be the entry in the ith row and jth column. Consider the set of $n \times n$ matrices over $\mathbb{F}$. Define

$$\det([a]) = a$$

and

$$\det(A) = \sum_{i=1}^{n}(-1)^{i+1}a_{i,1}\det(\hat{A}_{i,1})$$

5.1 Multilinear and Alternating

Definition 5.2. Let $V_1, \dots V_n$ be vector spaces over a field $\mathbb{F}$. The product $V_1 \times \dots \times V_n$ is defined by

$$V_1 \times \dots \times V_n = \{(v_1, \dots v_n) \ \mid \ v_1 \in V_1, \dots v_n \in V_n\}$$

Of course, with the appropriate operations $V_1 \times \dots \times V_n$ is a vector space.

Proposition 5.1. *$V_1 \times \dots \times V_n$ is a vector space over $\mathbb{F}$ with the following operations:*

$$(v_1, \dots, v_n) + (u_1, \dots, u_n) = (v_1 + u_1, \dots, v_n + u_n)$$

$$c(v_1, \dots, v_n) = (cv_1, \dots, cv_n).$$

The proof of the above proposition is standard, we will omit it.

Definition 5.3. Let $V_1, \dots, V_n, W$ be vector spaces over a field $\mathbb{F}$. A map $\varphi : V_1 \times \dots \times V_n \to W$ is called *multilinear* if for each fixed i and fixed elements $v_j \in V_j$, $j \neq i$, the map

$$V_i \to W \quad \text{defined by} \quad x \mapsto \varphi(v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_n)$$

is linear. If each $V_i = V$, $i = 1, 2, \dots n$ then φ is called a n-multilinear function on V. If W is a field, then φ is called a multilinear form on V.

The function $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is viewed as a multilinear map by viewing the columns of a matrix as column vectors and making the following identification.

$$M_{n\times n}(\mathbb{F}) \ni A = [v_1, \dots, v_n] \mapsto (v_1, \dots, v_n) \in \mathbb{F}^n \times \dots \times \mathbb{F}^n.$$

Proposition 5.2. $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is a multilinear function (viewing each matrix as a tuple of column vectors in $\mathbb{F}^n \times \dots \times \mathbb{F}^n$).*

Definition 5.4. An n-multilinear function φ on V is called alternating if φ is zero whenever two consecutive arguments are equal, i.e. if $v_i = v_{i+1}$ for some $i \in \{1, \dots, n-1\}$, then $\varphi(v_1, \dots, v_n) = 0$.

Lemma 5.5. *Let $B \in M_{n\times n}(\mathbb{F})$, where $n \geq 2$. If the column j of B equals*

$$e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} - \textit{ kth spot}$$

for some k $(1 \leq k \leq n)$, then

$$\det(B) = (-1)^{j+k} \det(\hat{B}_{k,j})$$

Theorem 5.6. *The determinant of a square matrix can be evaluated by cofactor expansion along any column or row, i.e.*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j})$$

or

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j})$$

Proposition 5.3. *The determinant function* $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is an alternating function.*

Theorem 5.7. *The determinant* $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is the* unique *multilinear alternating map taking the identity matrix to the mulitiplicative identity element in* $\mathbb{F}$

5.2 The Multiplicative Property

Definition 5.8 (Elementary Row Operations)**.** An elementary row operation is any one of the following operations performed on a matrix.

- Switching the position of two rows.
- Multiplying the entries of a row by a scalar.
- Replacing a row with its addition of a scalar multiple of another row.

Lemma 5.9. *Let A and B be matrices such that $C = AB$ is defined. Suppose $e_1, \ldots e_n$ be a sequence of elementary row operations. Let A' be the matrix resulting from performing $e_1, \ldots, e_n$ on A and C' be the matrix resulting from performing $e_1, \ldots, e_n$ on C. Then*

$$C' = A'B$$

Lemma 5.10. *Suppose $e_1, \dots e_n$ be a sequence of elementary row operations. Let A' be the matrix resulting from performing $e_1, \dots, e_n$ on A. We have that*

$$\alpha \det(A') = \det A$$

for some $\alpha \in \mathbb{F}$ depending only on $e_1, \dots, e_n$.

Definition 5.11. Let A be a matrix. Define its transpose, denoted $A^\top$, by

$$(A^\top)_{i,j} = A_{j,i}$$

Lemma 5.12. *Let A and B be matrices such that AB is defined.*

$$(AB)^\top = B^\top A^\top$$

Lemma 5.13. *For any square matrix A*

$$\det(A) = \det(A^\top)$$

Definition 5.14. A lower triangular matrix is any matrix L such that

$$L_{i,j} = 0 \quad \text{for} \quad j > i$$

Definition 5.15. An upper triangular matrix is any matrix U such that

$$U_{i,j} = 0 \quad \text{for} \quad j < i$$

Lemma 5.16. *The product of upper (lower) triangular matrices is an upper (lower) triangular matrix.*

Lemma 5.17. *Let A be either an upper triangular matrix or lower triangular matrix. We have that*

$$\det(A) = \prod a_{i,i}.$$

Proposition 5.4.

$$\det(AB) = \det(A)\det(B)$$

5.3 Invertibility of a Matrix

Definition 5.18. A matrix $A \in M_{n\times n}(\mathbb{F})$ is said to be invertible if there exists a matrix $B \in M_{n\times n}(\mathbb{F})$ such that

$$AB = I$$

$$BA = I$$

where I is the $n \times n$ identity matrix.

Definition 5.19. Suppose S is a set with a binary operation $\cdot : S \times S \to S$, then S with $\cdot$ is a monoid if

(a) For all a, b and c in S

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(b) There exists an identity element 1 in S such that for all $a \in S$

$$a \cdot 1 = 1 \cdot a = a.$$

Definition 5.20. Let A be a monoid and 1 its identity element. We say $a \in A$ is invertible if and only if there exists a $b \in A$ such that

$$a \cdot b = b \cdot a = 1$$

Definition 5.21. A monoid homomorphism between monoids A and B is a function $\phi : A \to B$ such that

$$\phi(a_1) \cdot \phi(a_2) = \phi(a_1 \cdot a_2)$$

for all $a_1, a_2 \in A$ and

$$\phi(1_A) = 1_B$$

for the identity elements $1_A \in A$ and $1_B \in B$.

Theorem 5.22. *Let A and B be two monoids and $\phi : A \to B$ a monoid homomorphism. If $a \in A$ is invertible then $\phi(a)$ is invertible.*

Theorem 5.23. *Let $T \in \mathcal{L}(V, W)$ and let $\beta = \{v_1, \dots v_n\}$, $\gamma = \{w_1, \dots w_m\}$ be bases for V and W respectively. The transformation T is invertible if and only if $[T]_\beta^\gamma$ is invertible.*

Definition 5.24. An elementary matrix is any matrix obtained from performing a single row operation on the identity matrix.

Proposition 5.5. *Let A be a matrix and let B be the matrix obtained from A by performing a row operation with corresponding elementary matrix E. We have*

$$EA = B$$

Proposition 5.6. *If A is a matrix and B is an upper triangular matrix obtained from A via a finite sequence of row operations, then A is invertible if and only if B is invertible. Moreover, B is invertible if and only if the entries on the diagonal are non-zero.*

Theorem 5.25. *A matrix A is invertible if and only if $\det(A) \neq 0$.*

5.4 Properties and Facts

- If A is a triangular matrix then $\det(A)$ is the product of the entries on the main diagonal.
- If a multiple of one row of A is added to another row to produce B then $\det(A) = \det(B)$.
- If two rows of A are interchanged to produce B then $\det(A) = -\det(B)$.
- If one row of A is multiplied by k to produce B then $\det(B) = k \cdot \det(A)$.
- A square matrix is invertible if and only if $\det(A) \neq 0$.
- $\det(A^\top) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(I) = 1$.

6 Eigenvalues and Eigenvectors

Note: We will assume all vector spaces in this section are complex! (However, a lot of the following is definable for a vector spaces over any field)

Definition 6.1. Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant under* T if $u \in U$ implies that $Tu \in U$.

Definition 6.2. Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of T if there exists a $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. The vector v is called an *eigenvector*.

Proposition 6.1. *$T \in \mathcal{L}(V)$ has a one dimensional subspace if and only if T has an eigenvalue.*

Theorem 6.3. *Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V)$. The following are equivalent:*

(a) $\lambda \in \mathbb{F}$ is an eigenvalue.

(b) $T - \lambda Id$ is not injective.

(c) $T - \lambda Id$ is not surjective.

(d) $T - \lambda Id$ is not invertible.

Proposition 6.2. *Suppose $T \in \mathcal{L}(V)$. If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors then $\{v_1, \ldots, v_m\}$ is a linearly independent set.*

Corollary 6.4. *Suppose $T \in \mathcal{L}(V)$ and $\dim(V) = n$. Then T has at most n distinct eigenvalues.*

Definition 6.5. Suppose $T \in \mathcal{L}(V)$. Define for positive m

$$T^m = \underbrace{T \circ \ldots \circ T}_{m \text{ times}}$$

$$T^0 = Id$$

and if T is invertible

$$T^{-m} = \underbrace{T^{-1} \circ \ldots \circ T^{-1}}_{m \text{ times}}$$

Definition 6.6. Suppose $T \in \mathcal{L}(V)$ and let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial over $\mathbb{F}$. Define

$$p(T) = a_n T^n + \ldots + a_1 T + a_0 Id$$

Proposition 6.3. *Let $T \in \mathcal{L}(V)$. If $p(x)$ is a polynomial over $\mathbb{F}$ and $p(x) = s(x)q(x)$ then $p(T) = s(T)q(T)$.*

Theorem 6.7. *Every operator on a finite dimensional, non-zero,* complex *vector space has an eigenvalue.*

6.1 Upper Triangular Matrices

Proposition 6.4. *Suppose $T \in \mathcal{L}(V)$ and $\beta = \{v_1, \ldots, v_n\}$ is a basis of V. The following statements are equivalent.*

(a) $[T]_\beta$ is upper triangular.

(b) $Tv_j \in \operatorname{span}\{v_1, \ldots, v_j\}$ *for all* $1 \leq j \leq n$*.*

(c) $\operatorname{span}\{v_1, \ldots, v_j\}$ *is invariant under* T *for all* $1 \leq j \leq n$*.*

Theorem 6.8. *Suppose* $T \in \mathcal{L}(V)$ *where* V *is finite dimensional. The operator* T *has an upper triangular matrix with respect to some basis.*

Proposition 6.5. *Suppose* $T \in \mathcal{L}(V)$ *and* $\beta = \{v_1, \ldots, v_n\}$ *is a basis of* V*. If* $[T]_\beta$ *is an upper triangular matrix then the following statements hold:*

(a) T *is invertible if and only if the diagonal entries of* $[T]_\beta$ *are non-zero.*

(b) The eigenvalues of T *(and* $[T]_\beta$*) are the diagonal entries of* $[T]_\beta$*.*

(c) $\det([T]_\beta)$ *is the product of the eigenvalues of* T*.*

(d) If γ *is another basis for* V *then* $\det([T]_\gamma)$ *is the product of the eigenvalues of* T*.*

6.2 Diagonal Matrices

Proposition 6.6. *Suppose* $T \in \mathcal{L}(V)$ *where* V *is* n*-dimensional. If* T *has* n *distinct eigenvalues then* T *has a diagonal matrix with respect to some basis of* V*.*

Theorem 6.9. *Suppose* $T \in \mathcal{L}(V)$ *where* V *is finite dimensional. Let* $\lambda_1, \ldots, \lambda_m$ *be the distinct eigenvalues of* T*. The following are equivalent.*

(a) T *has a diagonal matrix with respect to some basis.*

(b) V *has a basis of eigenvectors*

(c) There exists one dimensional subspaces $U_1, \ldots, U_n$ *of* V *where* $U_i = \operatorname{span}\{v_i\}$ $1 \leq i \leq n$ *and each* U_i *is invariant under* T *such that* $V = U_1 \oplus \ldots \oplus U_n$*.*

(d) $V = \operatorname{null}(T - \lambda_1 Id) \oplus \ldots \oplus \operatorname{null}(T - \lambda_m Id)$

(e) $\dim(V) = \dim(\operatorname{null}(T - \lambda_1 Id)) + \ldots + \dim(\operatorname{null}(T - \lambda_m Id))$

7 Inner Products

Note: We will assume all vector spaces in this section are either real or complex!

Definition 7.1. Let $V = \mathbb{C}^n$ (or $V = \mathbb{R}^n$). We define the *dot product* of two vectors $\vec{x} = [x_1, \ldots, x_n]^\top$ and $\vec{y} = [y_1, \ldots, y_n]^\top$ as

$$\vec{x} \cdot \vec{y} = x_1\bar{y}_1 + \ldots + x_n\bar{y}_n$$

where if $z = a + ib \in \mathbb{C}$ for $a, b \in \mathbb{R}$ then $\bar{z} = a - ib$. We call $\bar{z}$ the *complex conjugate* of z.

Definition 7.2. Let V be a complex vector space (or a real vector space). An inner product on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \qquad (\text{or } \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R})$$

such that

(a) $\langle v, v \rangle \geq 0$ for all $v \in V$.

(b) $\langle v, v \rangle = 0$ if and only if $v = 0$.

(c) $\langle \lambda v + u, w \rangle = \lambda \langle v, w \rangle + \langle u, w \rangle$ for all $\lambda \in \mathbb{C}$ (or $\mathbb{R}$) and $u, v, w \in V$.

(d) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ (or $\langle v, w \rangle = \langle w, v \rangle$).

A vector space V with an inner product defined on V is called an *inner product space*.

Note: Properties (3) and (4) say that the inner product is linear in the first variable and conjugate linear in the second variable. This is the common convention in mathematics. Physics uses the opposite convention on inner products, i.e. the inner product is conjugate linear in the first variable and linear in the second variable. This is mostly a matter of convenience for physicists as they use the “bra-ket” notation for inner-products. *In this course we will use the mathematics convention!*

Example 7.3. The following are inner product spaces.

(a) $\mathbb{C}^n$ (or $\mathbb{R}^n$) with the dot product.

(b) $\ell^2(\mathbb{C}) = \{\mathbf{a} = (a_0, a_1, \ldots) | \sum_{n=0}^{\infty} |a_i|^2 < \infty\}$ with $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=0}^{\infty} a_i \overline{b_i}$.

(c) $\mathcal{L}^2(\mathbb{R}, dx) = \left\{ f : \mathbb{R} \to \mathbb{C} \middle| \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \right\}$ with $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g}(x) \, dx$.

Definition 7.4. The *norm* or *length* of a vector v in an inner product space V is defined to be $\|v\| = \sqrt{\langle v, v \rangle}$.

Definition 7.5. If $z = a + ib \in \mathbb{C}$ define the *modulus* (or absolute value) of a complex number to be $|z| := \sqrt{a^2 + b^2}$.

Lemma 7.6. *For $v \in V$ and $a \in \mathbb{F}$ we have $\|av\| = |a|\|v\|$.*

7.1 Geometry

Definition 7.7. We say that two vectors v and u in an inner product space V are *perpendicular* or *orthogonal* if $\langle v, u\rangle = 0$. We say two vectors v and u are *parallel* if $u = av$ for some $a \in \mathbb{F}$.

Theorem 7.8 (Pythagorean Theorem)**.** *If u and v are orthogonal vectors in an inner product space V then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Theorem 7.9 (Triangle Inequality)**.** *If u and v are vectors in an inner product space V then*

$$\|u + v\| \leq \|u\| + \|v\|.$$

Theorem 7.10 (Parallelogram Law)**.** *If u and v are vectors in an inner product space V then*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

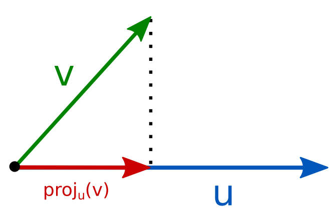
7.2 Orthogonal Projections

Definition 7.11. Given two vectors $\vec{v}$ and $\vec{u}$ in $\mathbb{R}^n$ we define

$$\hat{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right) \vec{u}$$

to be the *orthogonal projection* of $\vec{v}$ onto $\vec{u}$. This is sometimes denoted

$$\hat{v} = \text{proj}_{\vec{u}}(\vec{v}).$$



In the above picture the green vector $\vec{v}$ is being projected onto the blue vector $\vec{u}$ and the resulting red vector is the projection. Notice that the projection points in the same direction as $\vec{u}$.

Moreover,

$$\vec{z} = (\vec{v} - \hat{v}) \perp \vec{u}$$

and the length of $\vec{z} = \vec{v} - \hat{v}$ is represented by the dashed line.

Definition 7.12. Let $\vec{v}$ and $\vec{u}$ be two vectors let $\hat{v}$ be the projection of $\vec{v}$ onto $\vec{u}$. If we define $\vec{z} = \vec{v} - \hat{v}$ then

$$\vec{v} = \hat{v} + \vec{z}$$

where $\hat{v}$ is parallel to $\vec{u}$ and $\vec{z}$ is perpendicular to $\vec{u}$. We call $\hat{v}$ the component of $\vec{v}$ parallel to $\vec{u}$ and $\vec{z}$ is the component of $\vec{v}$ perpendicular to $\vec{u}$.

Definition 7.13. Let u and v be non-zero vectors in an inner product space V. The orthogonal decomposition of u with respect to v is defined to be

$$u = \frac{\langle u, v\rangle}{\|v\|^2}v + \left(u - \frac{\langle u, v\rangle}{\|v\|^2}v\right)$$

where $\frac{\langle u,v\rangle}{\|v\|^2}v$ is parallel to v and $\left(u - \frac{\langle u,v\rangle}{\|v\|^2}v\right)$ is perpendicular to v.

Theorem 7.14 (Cauchy-Schwarz Inequality)**.** *If u and v are vectors in an inner product space V then*

$$|\langle u, v\rangle| \leq \|u\| \ \|v\|.$$

7.3 Orthonormal Basis for a Vector Space

Definition 7.15. A list of vectors $\{v_1, \ldots, v_k\}$ is called *orthogonal* if

$$\langle v_i, v_j\rangle = 0 \qquad i \neq j$$

for all $i, j \in \{1, \ldots, k\}$. An *orthogonal basis* is a basis that is also an orthogonal list.

Definition 7.16. A list of vectors $\{v_1, \ldots, v_k\}$ is called *orthonormal* if

$$\langle v_i, v_j\rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

for all $i, j \in \{1, \ldots, k\}$. An *orthonormal basis* is a basis that is also an orthonormal list.

Proposition 7.1. *If $\{e_1, \ldots, e_m\}$ is an orthonormal list of vectors then*

$$\|a_1e_1 + \ldots + a_me_m\|^2 = |a_1|^2 + \ldots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$.

Corollary 7.17. *Every orthonormal list of vectors is linearly independent.*

Theorem 7.18. *If $\{v_1, \ldots, v_m\}$ is a linearly independent list of vectors in V then there exists an orthonormal list of vectors $\{e_1, \ldots, e_m\}$ of vectors in V such that span$\{v_1, \ldots, v_k\} = \operatorname{span}\{e_1, \ldots, e_k\}$ for $k \in \{1, \ldots, m\}$.*

Corollary 7.19. *Every finite dimensional inner product space has an orthonormal basis.*

Corollary 7.20. *Every orthonormal list of vectors in V can be extended to a orthonormal basis of V.*