Contents

1 Introduction and Definition

1.1 Introduction

Originally the use of determinants pre-dates the use of matrices and was viewed as a property of a system of equations. The determinant "determines" whether a system has a unique solution (this is of course related to whether the matrix associated to a system is invertible). Although the determinant has a wide variety of uses in geometry, representing the volume of a parallel-piped, algebraically one can view the determinant as a function with unique algebraic properties. A bit more formally, the determinant is a function

$$\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F},$$

where $\mathbb{F}$ is a field and $M_{n\times n}(\mathbb{F})$ is the set of matrices with entries from $\mathbb{F}$.

There are a variety of ways to define the determinant, see for instance: `https://en.wikipedia.org/wiki/Determinant#Definition`. For the time being, in this set of notes we will define the determinant via "cofactor expansion along the first column"

1.2 Definition of the Determinant

Definition 1. Let $\hat{A}_{i,j}$ be the $(n-1)\times(n-1)$ matrix that results from A by removing the ith row and jth column and let $a_{i,j}$ be the entry in the ith row and jth column. Consider the set of $n\times n$ matrices over $\mathbb{F}$. Define

$$\det([a]) = a$$

and

$$\det(A) = \sum_{i=1}^{n}(-1)^{i+1}a_{i,1}\det(\hat{A}_{i,1})$$

Example 1. For convenience note that the determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

As an example consider the following:

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\right) = 1\cdot 5 - 2\cdot 3 = -1$$

Example 2. Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

Solution:

$$\begin{aligned}\det(A) &= \sum_{i=1}^{n}(-1)^{i+1}a_{i,1}\det(\hat{A}_{i,1}) \\ &= (-1)^2\cdot 1\cdot\det\left(\begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix}\right) + (-1)^3\cdot 4\cdot\det\left(\begin{bmatrix} 5 & 0 \\ -2 & 0 \end{bmatrix}\right) + (-1)^4\cdot 0\cdot\det\left(\begin{bmatrix} 5 & 0 \\ 4 & -1 \end{bmatrix}\right) \\ =& -2\end{aligned}$$

As we mentioned in the introduction, the determinant is a function with some unique properties. In fact, the following theorem is true.

Theorem 1. *The determinant* $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is the* unique *multilinear alternating map taking the identity matrix to the mulitiplicative identity element in* $\mathbb{F}$

The proof that the determinant is the *only* map with this property is not necessary for this course. In these notes we will prove the map det is both multilinear and alternating (to be defined in Section 2). The proof that the determinant takes the identity matrix to the multiplicative identity in the field is relatively easy to see from the definition and we will omit the proof. Moreover, one should notice that Theorem 1 shows that the various versions of the definition of the determinant are equivalent as long as they have the correct properties.

2 Multilinear and Alternating

In this section we will prove that the map det is both multilinear and alternating. We will need the following definition and proposition.

Definition 2. Let $V_1, \dots V_n$ be vector spaces over a field $\mathbb{F}$. The product $V_1 \times \dots \times V_n$ is defined by

$$V_1 \times \dots \times V_n = \{(v_1, \dots v_n) \;\mid\; v_1 \in V_1, \dots v_n \in V_n\}$$

Of course, with the appropriate operations $V_1 \times \dots \times V_n$ is a vector space.

Proposition 2. *$V_1 \times \dots \times V_n$ is a vector space over $\mathbb{F}$ with the following operations:*

$$(v_1, \dots, v_n) + (u_1, \dots, u_n) = (v_1 + u_1, \dots, v_n + u_n)$$

$$c(v_1, \dots, v_n) = (cv_1, \dots, cv_n).$$

The proof of the above proposition is standard, we will omit it.

2.1 Multilinearity

Definition 3. Let $V_1, \dots, V_n, W$ be vector spaces over a field $\mathbb{F}$. A map $\varphi : V_1 \times \dots \times V_n \to W$ is called *multilinear* if for each fixed i and fixed elements $v_j \in V_j$, $j \neq i$, the map

$$V_i \to W \quad \text{defined by} \quad x \mapsto \varphi(v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_n)$$

is linear. If each $V_i = V$, $i = 1, 2, \dots n$ then φ is called a n-multilinear function on V. If W is a field, then φ is called a multilinear form on V.

The function $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is viewed as a multilinear map by viewing the columns of a matrix as column vectors and making the following identification.

$$M_{n\times n}(\mathbb{F}) \ni A = [v_1, \dots, v_n] \mapsto (v_1, \dots, v_n) \in \mathbb{F}^n \times \dots \times \mathbb{F}^n.$$

Before we prove the determinant is a multilinear function we will show an example of the multilinear property

Example 3. Let $k \in \mathbb{F}$.

$$\det\left(\begin{bmatrix} 2+k3 & 1 \\ 5+k2 & 2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}\right) + k\det\left(\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\right)$$

Proposition 3. $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is a multilinear function (viewing each matrix as a tuple of column vectors in* $\mathbb{F}^n \times \ldots \times \mathbb{F}^n$*).*

Proof. We proceed by induction on n. The result is clear for $n = 1$, so suppose that for an integer $n \geq 2$ the determinant of any $(n-1)\times(n-1)$ matrix is a linear function of each column when the remaining columns are fixed. Suppose that $a_r = u + kv$ for some r, where $u, v \in \mathbb{F}^n$. Let A be a matrix and $a_1, \ldots, a_n$ be the columns of A. We want to show that

$$\det(a_1 \ldots a_{r-1}, u+kv, a_{r+1}, \ldots a_n) = \det(a_1 \ldots a_{r-1}, u, a_{r+1}, \ldots a_n) + k\det(a_1 \ldots a_{r-1}, v, a_{r+1}, \ldots a_n).$$

Let

$$u = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Let B and C be the the matrices obtained by replacing column a_r with u and v respectively. We must then prove

$$\det(A) = \det(B) + k\det(C).$$

We leave the proof to the reader for the case that $a_r = a_1$. For $r > 1$ and $1 \leq i \leq n$ the columns of $\hat{A}_{i,1}$, $\hat{B}_{i,1}$, and $\hat{C}_{i,1}$ are the same except for column (now labeled) $r-1$. Moreover, column $r-1$ of $\hat{A}_{i,1}$ is

$$\begin{bmatrix} b_1 + kc_1 \\ \vdots \\ b_{i-1} + kc_{i-1} \\ b_{i+1} + kc_{i+1} \\ \vdots \\ b_n + kc_n \end{bmatrix}$$

which is the sum of column $r-1$ of $\hat{B}_{i,1}$ and k times column $r-1$ of $\hat{C}_{i,1}$. Since $\hat{B}_{i,1}$ and

$\hat{C}_{i,1}$ are $(n-1) \times (n-1)$ by the induction hypothesis

$$\det(\hat{A}_{i,1}) = \det(\hat{B}_{i,1}) + k \det(\hat{C}_{i,1}).$$

Then

$$\begin{aligned}
\det(A) &= \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(\hat{A}_{i,1}) \\
&= \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} (\det(\hat{B}_{i,1}) + k \det(\hat{C}_{i,1})) \\
&= \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(\hat{B}_{i,1}) + k \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(\hat{C}_{i,1}) \\
&= \det(B) + k \det(C)
\end{aligned}$$

□

2.2 Alternating

Definition 4. An n-multilinear function φ on V is called alternating if φ is zero whenever two consecutive arguments are equal, i.e. if $v_i = v_{i+1}$ for some $i \in \{1, \ldots, n-1\}$, then $\varphi(v_1, \ldots, v_n) = 0$.

For ease, we will prove that the determinant can be calculated via cofactor expansion along any column. In fact, the determinant can be calculated via cofactor expansion along any row as well. To prove this we will need the following technical lemma and theorem.

Lemma 4. *Let $B \in M_{n \times n}(\mathbb{F})$, where $n \geq 2$. If the column j of B equals*

$$e_k = \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ - \textit{ kth spot} \\ \\ \\ \end{array} \end{array}$$

for some k $(1 \leq k \leq n)$, then

$$\det(B) = (-1)^{j+k} \det(\hat{B}_{k,j})$$

Proof. We will proceed by induction on n. Assume $n \geq 3$ and that for $(n-1) \times (n-1)$ matrices the lemma is true. Let B be an $n \times n$ matrix whose jth row is equal to e_k for some

$1 \le k \le n$. The result follows immediately from the definition if $j = 1$. Suppose therefore that $1 < j \le n$. For each $i \neq k$, $(1 \le i \le k)$, Let $C_{i,j}$ denote the $(n-2) \times (n-2)$ matrix obtained from B by deleting column 1 and j and rows i and k. Notice, that for each i column $j-1$ of $\hat{B}_{i,1}$ is either

$$\begin{cases} e_{k-1} & \text{if } i < k \\ 0 & \text{if } i = k \\ e_k & \text{if } i > k \end{cases}$$

By the induction hypothesis,

$$\det \hat{B}_{i,1} = \begin{cases} (-1)^{(j-1)+(k-1)} \det(C_{i,j}) & \text{if } i < k \\ 0 & \text{if } i = k \\ (-1)^{(j-1)+k} \det(C_{i,j}) & \text{if } i > k \end{cases}$$

Therefore,

$$\begin{aligned} \det(B) &= \sum_{i=1}^{n} (-1)^{i+1} b_{i,1} \det(\hat{B}_{i,1}) \\ &= \sum_{i<k} (-1)^{i+1} b_{i,1} \det(\hat{B}_{i,1}) + \sum_{i>k} (-1)^{i+1} b_{i,1} \det(\hat{B}_{i,1}) \\ &= \sum_{i<k} (-1)^{i+1} b_{i,1} (-1)^{(j-1)+(k-1)} \det(C_{i,j}) + \sum_{i>k} (-1)^{i+1} b_{i,1} (-1)^{(j-1)+k} \det(C_{i,j}) \\ &= (-1)^{j+k} \left[\sum_{i<k} (-1)^{i+1} b_{i,1} \det(C_{i,j}) + \sum_{i>k} (-1)^{(i-1)+1} b_{i,1} \det(C_{i,j}) \right] \\ &= (-1)^{j+k} \det(\hat{B}_{k,j}) \end{aligned}$$

Therefore by induction we have that the result is true for all n. $\square$

Theorem 5. *The determinant of a square matrix can be evaluated by cofactor expansion along any column, i.e.*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j})$$

Proof. Let column j of the matrix A be written as $\sum_{i=1}^{n} a_{i,j} e_i$. Let B_i denote the matrix obtained from A by replacing column j by e_i. Thus by the multilinearity property

$$\det(A) = \sum_{i=1^n} a_{i,j} \det(B_i) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j}).$$

$\square$

Proposition 6. *The determinant function* $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is an alternating function.*

Proof. We proceed by induction. We leave the reader to prove the base case of $n \leq 2$. Let $n \geq 3$, and assume that for all matrices of size $(n-1)\times(n-1)$, that if the matrix has two identical columns the determinant is zero. Suppose column r and s of A are identical and that $a_l \neq a_m$ for $r \neq s$. Since,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j})$$

we can expand on any column that is not r or s. Then $\det(\hat{A}_{i,j}) = 0$ for each i and j since $\hat{A}_{i,j}$ is of size $(n-1)\times(n-1)$ and contains two identical columns. $\square$

3 Other Algebraic Properties

3.1 The Determinant Redefined

As stated in the beginning of section 2.2 the determinant can be calculated by cofactor expansion along any row or column. One can replicate the above proofs and theorems with rows replacing columns. We now make this the formal definition for the determinant.

Definition 5. Let $\hat{A}_{i,j}$ be the $(n-1)\times(n-1)$ matrix that results from A by removing the ith row and jth column and let $a_{i,j}$ be the entry in the ith row and jth column. Consider the set of $n \times n$ matrices over $\mathbb{F}$. Define

$$\det([a]) = a$$

and

$$\begin{aligned} \det(A) &= \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j}) \\ &= \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j}) \end{aligned}$$

3.2 The Multiplicative Property

In this section we will prove the multiplicative property of the determinant, i.e. the determinant of a product is the product of the determinants. We begin by listing out a few definitions and lemmas we will need for the main result. While not incredibly difficult, the proofs will either be omitted or sketched.

Definition 6 (Elementary Row Operations)**.** An elementary row operation is any one of the following operations performed on a matrix.

- Switching the position of two rows.
- Multiplying the entries of a row by a scalar.
- Replacing a row with its addition of a scalar multiple of another row.

Lemma 7. *Let A and B be matrices such that $C = AB$ is defined. Suppose $e_1, \dots e_n$ be a sequence of elementary row operations. Let A' be the matrix resulting from performing $e_1, \dots, e_n$ on A and C' be the matrix resulting from performing $e_1, \dots, e_n$ on C. Then*

$$C' = A'B$$

Lemma 8. *Suppose $e_1, \dots e_n$ be a sequence of elementary row operations. Let A' be the matrix resulting from performing $e_1, \dots, e_n$ on A. We have that*

$$\alpha \det(A') = \det A$$

for some $\alpha \in \mathbb{F}$ depending only on $e_1, \dots, e_n$.

Definition 7. Let A be a matrix. Define its transpose, denoted $A^\top$, by

$$(A^\top)_{i,j} = A_{j,i}$$

Lemma 9. *Let A and B be matrices such that AB is defined.*

$$(AB)^\top = B^\top A^\top$$

Lemma 10. *For any square matrix A*

$$\det(A) = \det(A^\top)$$

Proof. Note that cofactor expansion along any row is equivalent to cofactor expansion along any column. $\square$

Definition 8. A lower triangular matrix is any matrix L such that

$$L_{i,j} = 0 \quad \text{for} \quad j > i$$

Definition 9. An upper triangular matrix is any matrix U such that

$$U_{i,j} = 0 \quad \text{for} \quad j < i$$

Lemma 11. *The product of upper (lower) triangular matrices is an upper (lower) triangular matrix.*

Lemma 12. *Let A be either an upper triangular matrix or lower triangular matrix. We have that*

$$\det(A) = \prod a_{i,i}.$$

Proof. Proceed by induction and using cofactor expansion along the first column. □

We are now in a position to prove one of our main propositions.

Proposition 13.

$$\det(AB) = \det(A)\det(B)$$

Proof. Let A and B be $n \times n$ matrices and $C = AB$. The matrix A can be turned into an upper triangular matrix by a finite sequence of row operations $e_1, \ldots e_n$. Let A' be the upper triangular matrix resulting from performing $e_1, \ldots e_n$ on A. Let C' be the the matrix resulting from performing $e_1, \ldots, e_n$ on C. Note,

$$C' = A'B$$

by Lemma 7. By Lemma 8 there exists an $\alpha \in \mathbb{F}$ such that

$$\alpha \det(A') = \det(A)$$

and

$$\alpha \det(C') = \det(C).$$

In addition,

$$(C')^\top = B^\top (A')^\top.$$

The matrix $B^\top$ by a sequence of elementary row operations $f_1, \ldots, f_m$. Let $(B^\top)'$ be the upper triangular matrix resulting from performing $f_1, \ldots f_m$ on $B^\top$. Let C'' be the the matrix resulting from performing $f_1, \ldots, f_m$ on $(C')^\top$. So

$$C'' = (B^\top)'(A')^\top$$

and there exists a $\beta \in \mathbb{F}$ such that

$$\beta \det((B^\top)') = \det(B^\top)$$

and

$$\beta \det(C'') = \det((C')^\top).$$

The product of lower triangular matrices is a lower triangular matrix. Hence, $(B^\top)'(A')^\top$ is a lower triangular matrix. By an application of Lemma 12

$$\det((B^\top)'(A')^\top) = \det((B^\top)') \det((A')^\top).$$

Using an application of Lemma 10

$$\begin{aligned}
\det(C) &= \alpha \det(C') \\
&= \alpha \det((C')^\top) \\
&= \alpha\beta \det(C'') \\
&= \alpha\beta \det((B^\top)'(A')^\top) \\
&= \alpha\beta \det((B^\top)') \det((A')^\top) \\
&= \beta \det((B^\top)')\alpha \det((A')^\top) \\
&= \det(A) \det(B)
\end{aligned}$$

□

3.3 Invertibility of a Matrix

We begin with a definition.

Definition 10. A matrix $A \in M_{n\times n}(\mathbb{F})$ is said to be invertible if there exists a matrix $B \in M_{n\times n}(\mathbb{F})$ such that

$$AB = I$$

$$BA = I$$

where I is the $n \times n$ identity matrix.

In an elementary linear algebra course one learns that a matrix is invertible if and only if the determinant is non-zero. First we will show that the multiplicative property of the determinant means that invertible matrices are taken to invertible elements in the field.

As a small detour we show essentially any function with the multiplicative property takes invertible elements to invertible elements.

3.3.1 Monoid Homomorphisms

Definition 11. Suppose S is a set with a binary operation $\cdot : S \times S \to S$, then S with $\cdot$ is a monoid if

(a) For all a, b and c in S
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(b) There exists an identity element 1 in S such that for all $a \in S$
$$a \cdot 1 = 1 \cdot a = a.$$

Definition 12. Let A be a monoid and 1 its identity element. We say $a \in A$ is invertible if and only if there exists a $b \in A$ such that
$$a \cdot b = b \cdot a = 1$$

Definition 13. A monoid homomorphism between monoids A and B is a function $\phi : A \to B$ such that
$$\phi(a_1) \cdot \phi(a_2) = \phi(a_1 \cdot a_2)$$
for all $a_1, a_2 \in A$ and
$$\phi(1_A) = 1_B$$
for the identity elements $1_A \in A$ and $1_B \in B$.

Theorem 14. *Let A and B be two monoids and $\phi : A \to B$ a monoid homomorphism. If $a \in A$ is invertible then $\phi(a)$ is invertible.*

Proof. Suppose a is invertible and e is an inverse. Thus
$$ae = 1_A = ea$$
Apply ϕ,
$$\phi(ae) = \phi(a)\phi(e) = 1_B = \phi(e)\phi(a) = \phi(ea).$$
$\square$

We now can prove the following.

Theorem 15. *If A is an invertible matrix then* $\det(A) \neq 0$.

Proof. It is clear from the definitions that both $n \times n$ matrices over a field $\mathbb{F}$ and the field $\mathbb{F}$ itself are both monoids (with their standard multiplications). Moreover, the determinant is a monoid homomorphism since it has the multiplicative property. $\square$

In the remainder of this section we will show that if the determinant of a matrix is non-zero then the matrix is invertible. To do so we will use the fact that the inveritibility of a matrix A is tied to the invertibility of the transformation $T : \mathbb{F}^n \to \mathbb{F}^n$ given by

$$T(x) = A \cdot x.$$

More explicitly we recall the following theorem from class.

Theorem 16. *Let $T \in \mathcal{L}(V, W)$ and let $\beta = \{v_1, \dots v_n\}$ and $\gamma = \{w_1, \dots w_m\}$ be ordered bases for V and W respectively. There exists a matrix representation with respect to β and γ*

$$A = [T]_\beta^\gamma$$

defined by

$$Tv_k = A_{1,k}w_1 + \dots + A_{m,k}w_m \quad \text{for } k \in \{1, \dots n\}$$

such that the mapping $M : \mathcal{L}(V, W) \to \mathbb{F}^{m \times n}$ defined by

$$M(T) = [T]_\beta^\gamma$$

is a vector space isomorphism.

This matrix is unique for a given choice of β and γ. Moreover, we have the following multiplicative property.

Theorem 17. *Let $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, X)$. Let $\beta = \{v_1, \dots v_n\}$, $\gamma = \{w_1, \dots w_m\}$ and $\delta = \{x_1, \dots, x_p\}$ be ordered bases for V, W and X respectively. We have*

$$[ST]_\beta^\delta = [S]_\gamma^\delta [T]_\beta^\gamma$$

This leads to the following theorem.

Theorem 18. *Let $T \in \mathcal{L}(V, W)$ and let $\beta = \{v_1, \dots v_n\}$, $\gamma = \{w_1, \dots w_m\}$ be bases for V and W respectively. The transformation T is invertible if and only if $[T]_\beta^\gamma$ is invertible.*

Proof. Assume T in invertible, then

$$I = [\mathrm{Id}_V]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$$

and

$$I = [\mathrm{Id}_W]_\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta.$$

Conversely, assume $[T]_\beta^\gamma$ is invertible. Then there is a matrix B such that

$$I = B[T]_\beta^\gamma = [T]_\beta^\gamma B$$

Since M is an isomorphism, there exists a linear transformation S such that $[S]_\gamma^\beta = B$. We must have that

$$S = T^{-1}$$

from the above equations. $\square$

With Theorem 18 in mind, we note a matrix B is invertible if and only if the transformation $T : \mathbb{F}^n \to \mathbb{F}^n$ given by $T(x) = Bx$ is invertible. We now need the following definition and proposition. We will omit the propositions proof since its simple.

Definition 14. An elementary matrix is any matrix obtained from performing a single row operation on the identity matrix.

Proposition 19. *Let A be a matrix and let B be the matrix obtained from A by performing a row operation with corresponding elementary matrix E. We have*

$$EA = B$$

Note this means that the elementary matrices are invertible. Moreover, the following proposition will allow us to reduce our remaining result to a relatively simple case.

Proposition 20. *If A is a matrix and B is an upper triangular matrix obtained from A via a finite sequence of row operations, then A is invertible if and only if B is invertible. Moreover, B is invertible if and only if the entries on the diagonal are non-zero.*

Proof. Suppose A is a matrix and B is an upper triangular matrix obtained from performing a sequence of row operations on A, i.e. we have that

$$E_k \cdots E_2 E_1 A = B$$

for some finite sequence of elementary matrices $E_1, \ldots, E_k$. Since each of the elementary matrices are invertible we see A is invertible if and only if B is invertible. If B is invertible and upper triangular then diagonal entries are non-zero otherwise the transformation $T : \mathbb{F}^n \to \mathbb{F}^n$ given by

$$T(x) = Bx$$

is not injective. If B is upper triangular and has non-zero entries, then the range of T has dimension n. Hence T is surjective and thus bijective. Thus, B is invertible as a matrix. $\square$

We are now in a position to prove the following.

Theorem 21. *If A is a matrix and $\det(A) \neq 0$ then A is invertible.*

Proof. Suppose $\det(A) \neq 0$. Let B be an upper triangular matrix obtained from A via row operations. There exist a finite sequence of elementary matrices such that

$$E_k \cdots E_2 E_1 A = B.$$

If A is not invertible, then B is also not invertible. Hence, B has a zero entry on its diagonal and the determinant of B is zero by Lemma 12. Thus the determinant of A is zero as well. This is a contradiction on our hypothesis. $\square$

In reality, the above proof can be turned into an if and only if proof and the detour into monoid homomorphisms is a bit unnecessary. But combining Theorems 15 and 21 we have the main theorem of this section.

Theorem 22. *A matrix A is invertible if and only if $\det(A) \neq 0$.*

4 Appendix

In this section we provide a list of properties and facts.

Properties and Facts:

- If A is a triangular matrix then $\det(A)$ is the product of the entries on the main diagonal.
- If a multiple of one row of A is added to another row to produce B then $\det(A) = \det(B)$.
- If two rows of A are interchanged to produce B then $\det(A) = -\det(B)$.
- If one row of A is multiplied by k to produce B then $\det(B) = k \cdot \det(A)$.

Determinants

- A square matrix is invertible if and only if $\det(A) \neq 0$.
- $\det(A^\top) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(I) = 1$.