1. In this question I will guide you through the proof of the following proposition:

Proposition 0.1. *If $y \geq 0$ and $n \in \mathbb{N}^+$ then there is a unique non-negative real number s such that $s^n = y$. In particular, the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^n$ is one to one and onto.*

a) Prove the following two lemmas that are used throughout the rest of the proof.

Lemma 0.2. *If $x < y$ then $xz < yz$ for all $x, y, z \in \mathbb{R}^+$*

Lemma 0.3. *If $0 < x < 1$ then $x^n < x$ for all $n \in \mathbb{N}^+$ and if $x > 1$ then $x^n > x$ for all $n \in \mathbb{N}^+$.*

b) Let $S = \{x \in \mathbb{R} : x > 0 \text{ and } x^n < y\}$, $\alpha = \min\{1, y\}$ and $\beta = \max\{1, y\}$. Show that if $x < \alpha$ then $x \in S$ and that β is an upper-bound for S. From this information show that S has a least upper bound, call it s. We will show that this is the root for y.

c) Show that if $t > 0$ and $y < t^n$, then t is an upper bound for S.

d) Show that if $t > 0$ and $y < t^n$, then there is a $v \in (0, t)$ such that $y < v^n$. Hence, $v < t$ and v is an upper bound for S. In particular, t is not the least upper bound for S and therefore $s^n \leq y$.
Hint: Suppose for the sake of contradiction that $v^n < y$ for all $v \in (0, t)$. Thus, $v^n < y < t^n$ and hence $0 < |t^n - y| < |t^n - v^n|$ (why?). Then conclude that this is a contradiction based on the fact that for a fixed t and $v \in (0, t)$

$$|t^n - v^n| \leq |t - v| \cdot M \text{ for some } M \in \mathbb{R}^+$$

(state why this is contradiction).

e) Show that if $t > 0$ and $t^n < y$ then there exists a v such that $0 < t < v$ and $v^n < y$. Use this to conclude that t is not an upper bound for S and thus $s^n \geq y$. In combination with the above parts conclude that $s^n = y$